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On the Δ-interval and the Δ-convexity numbers of graphs and graph products ☆

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Abstract

Given a graph G and a set $S \subseteq V(G)$, the Δ-interval of S, $[S]_\Delta$, is the set formed by the vertices of S and every $w \in V(G)$ forming a triangle with two vertices of S. If $[S]_\Delta = S$, then S is Δ-convex of G; if $[S]_\Delta = V(G)$, then S is a Δ-interval set of G. The Δ-interval number of G is the minimum cardinality of a Δ-interval set and the Δ-convexity number of G is the maximum cardinality of a proper Δ-convex subset of $V(G)$. In this work, we show that the problem of computing the Δ-convexity number is W[1]-hard and NP-hard to approximate within a factor $O\left(n^{1-\varepsilon}\right)$ for any constant $\varepsilon > 0$ even for graphs with diameter 2 and that the problem of computing the Δ-interval number is NP-complete for general graphs. For the positive side, we present characterizations that lead to polynomial-time algorithms for computing the Δ-convexity number of chordal graphs and for computing the Δ-interval number of block graphs. We also present results on the Δ-hull, Δ-interval and Δ-convexity numbers concerning the three standard graph products, namely, the Cartesian, the strong and the lexicographic products, in function of these and well-studied parameters of the operands.

Previous Next

Keywords

, Δ,-convexity; , Δ,-convexity number; , Δ,-hull number; , Δ,-interval number; Graph products

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A note on fold thickness of graphs

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Abstract

A 1-fold of G is the graph G' obtained from a graph G by identifying two nonadjacent vertices in G having at least one common neighbor and reducing the resulting multiple edges to simple edges. A uniform k-folding of a graph G is a sequence of graphs $G = G_0, G_1, G_2, \ldots, G_k$, where G_{i+1} is a 1-fold of G_i for $i = 0, 1, 2, \ldots, k-1$ such that all graphs in the sequence are singular or all of them are nonsingular. The largest k for which there exists a uniform k- folding of G is called fold thickness of G and this concept was first introduced in [1]. In this paper, we determine fold thickness of corona product graph $G \odot \overline{K_m}$, $G \odot_S \overline{K_m}$ and graph join $G + \overline{K_m}$.

Key Words: *Fold thickness, Uniform folding, Singular graphs.*

2020 AMS Subject Classification: *05C50, 05C76.*

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DECOMPOSITION DIMENSION OF SOME CLASS OF TREES

REJI T AND RUBY R

ABSTRACT. For an ordered k-decomposition $\mathscr{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph $G = (V, E)$, the $\mathscr{D}$-representation of an edge e is the k-tuple

$$\gamma(e/\mathscr{D}) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k)),$$

where $d(e, G_i)$ represents the distance from e to G_i. A decomposition $\mathscr{D}$ is resolving if every two distinct edges of G have distinct representations. The minimum k for which G has a resolving k-decomposition is its decomposition dimension dec(G). In this paper, the decomposition dimension of broom graph, double broom graph and upper bounds for the decomposition dimension of banana tree graph and fire cracker graph are determined.

1. INTRODUCTION

Let $G = (V, E)$ be a finite undirected connected graph without loops or multiple edges. A decomposition of a graph G is a collection of subgraphs of G, none of which have isolated vertices, whose edge sets provide a partition of $E(G)$. A decomposition of G into k subgraphs is a k-decomposition of G. A decomposition $\mathscr{D} = \{G_1, G_2, \ldots, G_k\}$ is ordered if the ordering $(G_1, G_2, \ldots, G_k)$ has been imposed on $\mathscr{D}$. If each subgraph G_i of $\mathscr{D}$ is isomorphic to a graph H, then $\mathscr{D}$ is said to be an H-decomposition of G.

For edges $e, f \in E(G)$, the distance $d(e, f)$ between e and f is the minimum non negative integer k for which there exists a sequence $e = e_0, e_1, e_2, \ldots, e_k = f$ of edges of G such that e_i and e_{i+1} are adjacent for $i = 0, 1, \ldots, k-1$. For an edge e of G and a subgraph F of G, $d(e, F) = \min\{d(e, f), f \in E(F)\}$. The following definitions are from [1]. Let $\mathscr{D} = \{G_1, G_2, \ldots, G_k\}$ be an ordered k-decomposition of G. The $\mathscr{D}$-representation of an edge e is the k-tuple $\gamma(e/\mathscr{D}) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k))$, where $d(e, G_i)$ represents the distance from e to G_i. We call $\mathscr{D}$ a resolving k-decomposition if for any pair of edges e and f, there exists some index i such that $d(e, G_i) \neq d(f, G_i)$. The minimum k for which G has a resolving k-decomposition is its decomposition dimension dec(G).

2. PRELIMINARIES

G. Chartrand *et al.* introduced these concepts in [1]. It is further studied in [3–5, 8]. The concepts of resolving set and minimum resolving set have appeared in the literature previously. Slater introduced and studied these ideas with a different terminology 'locating set' in [9] and [10]. Harary and Melter [6] discovered these concepts independently and these

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Least Common Multiple of Path, Star with Cartesian Product of Some Graphs

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Abstract A graph G without isolated vertices is a least common multiple of two graphs H_1 and H_2 if G is a smallest graph, in terms of number of edges, such that there exists a decomposition of G into edge disjoint copies of H_1 and H_2. The collection of all least common multiples of H_1 and H_2 is denoted by $\mathrm{LCM}(H_1, H_2)$ and the size of a least common multiple of H_1 and H_2 is denoted by $\mathrm{lcm}(H_1, H_2)$. In this paper $\mathrm{lcm}(P_4, P_m \square P_n)$, $\mathrm{lcm}(P_4, C_m \square C_n)$ and $\mathrm{lcm}(K_{1,3}, K_{1,m} \square K_{1,n})$ are determined.

Keywords graph decomposition; least common multiple

MR(2020) Subject Classification 05C38; 05C51; 05C70

1. Introduction

All graphs considered in this paper are assumed to be simple and to have no isolated vertices. The number of vertices of a graph G denoted by $v(G)$, is called the order of G and the number of edges of G denoted by $e(G)$, is called the size of G.

A graph H is said to divide a graph G if there exists a set of subgraphs of G, each isomorphic to H, whose edge sets partition the edge set of G. Such a set of subgraphs is called an H-decomposition of G. If G has an H-decomposition, we say that G is H-decomposable and write $H|G$.

A graph is called a common multiple of two graphs H_1 and H_2 if both $H_1|G$ and $H_2|G$. A graph G is a least common multiple of H_1 and H_2 if G is a common multiple of H_1 and H_2 and no other common multiple has fewer edges. Several authors have investigated the problem of finding least common multiples of pairs of graphs H_1 and H_2; that is graphs of minimum size which are both H_1 and H_2 decomposable. The problem was introduced by Chartrand et al. in [1] and they showed that every two nonempty graphs have a least common multiple. The problem of finding the size of least common multiples of graphs has been studied for several pairs of graphs: cycles and stars [1–3], paths and complete graphs [4], pairs of complete graphs, complete graphs and a 4-cycle, paths and stars and pairs of cycles. Least common multiple of digraphs were considered in [5].

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Projective Dimension of Some Graphs

REJI THANKACHAN, RUBY ROSEMARY and SNEHA BALAKRISHNAN

ABSTRACT. In this paper exact values for the projective dimension of edge ideals associated to some star related graphs and product graphs $G \,\square\, P_2$, when $G = C_n,\ K_n$ and upper bounds for the projective dimension when $G = P_n,\ W_n$, are obtained. We have proved that $pd(C_{n+1} \,\square\, P_2) = 2(n - \lfloor \frac{n}{4} \rfloor)$, $pd(K_n \,\square\, P_2) = 2n - 2$ and $pd(P_{n+1} \,\square\, P_2) \leq n + 3 + \lfloor \frac{n-3}{2} \rfloor$, $pd(W_n \,\square\, P_2) \leq n + 1 + \lceil \frac{2n-1}{3} \rceil$. These values are functions of the number of vertices in the corresponding graphs.

1. INTRODUCTION

In this paper all graphs are finite and simple. Let $V(G)$ denote the vertex set of a graph G and let (u, v) denote an edge of G with end points u and v. For $v \in V(G)$, let $N(v)$ denote the set of all vertices adjacent to v, called the neighbor set of G and $N[v] = N(v) \cup \{v\}$. Let S_n denote the star on $n + 1$ vertices $\{u_0, u_1, \ldots, u_n\}$ where u_0 is adjacent to all other vertices. The wheel graph W_n on $n + 1$ vertices is a graph obtained by connecting all n vertices of the cycle C_n to an $n + 1$-th vertex (called the hub). The edges connecting the hub and the vertices of C_n are called spokes.

The Cartesian product of two graphs G and H is denoted as $G \,\square\, H$. It is a graph with vertex set $V(G) \times V(H) = \{(g, h) | g \in G, h \in H\}$ and two vertices (g, h) and (g', h') are adjacent if and only if $g = g'$ and $hh' \in E(H)$ or $gg' \in E(G)$ and $h = h'$.

Let G is a graph with vertex set $V = \{x_1, x_2, \ldots, x_n\}$ and let $S = K[x_1, x_2, \ldots, x_n]$ be the polynomial ring over the field K. The edge ideal of G is the monomial ideal $I(G) \subseteq S$ generated by $\{x_i x_j : (x_i, x_j) \text{ is an edge of } G\}$. The edge ring of G is the quotient ring $S/I(G)$ [4]. Villarreal introduced the concept of edge ideal of a graph in [6].

Let $U = \{x_1, x_2, \ldots, x_n\}$ be a finite set. A simplicial complex Δ over U is a subset of the powerset U with the property that $\{v_1\}, \{v_2\}, \ldots, \{v_n\}$ belongs to Δ and if $F \in \Delta$ and $J \subseteq F$, then $J \in \Delta$. The elements of Δ are called faces and dimension of a face, $dim\ F = |F| - 1$. The dimension of the simplicial complex Δ, $dim\ \Delta$ is the maximum of the dimensions of its faces [4]. Associated to the edge ideal $I(G)$ of G is its independence complex, $ind(G)$, the simplicial complex on the vertex set V of G which has faces $\{\{x_{i_1}, x_{i_2}, \ldots, x_{i_m}\} |\ no\ \{x_{i_j}, x_{i_k}\}\ is\ an\ edge\ of\ G\}$ [3].

The Betti number of an ideal can be defined in terms of its $Stanley - Reisner\ complex$ using the Hochster's Formula.

Theorem 1.1. [3] *Let Δ be the Stanley-Reisner complex of a squarefree monomial ideal $I \subseteq S$ and let $\beta_{i,m}(I)$, where m is a squarefree monomial of degree greater than or equal to i, be the multigraded betti number of I. Then $\beta_{i-1,m}(I) = dim_K \tilde{H}_{deg\ m-i-1}(\Delta_m; K)$, where Δ_m is the subcomplex of Δ consisting of those faces whose vertices correspond to variables occuring in m and $\tilde{H}_k(\Delta)$ is the associated homology group of Δ.*

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Decomposition dimension of Cartesian product of some graphs

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For an ordered k-decomposition $\mathscr{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph $G = (V, E)$, the $\mathscr{D}$-representation of an edge e is the k-tuple $\gamma(e/\mathscr{D}) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k))$, where $d(e, G_i)$ represents the distance from e to G_i. A decomposition $\mathscr{D}$ is resolving if every two distinct edges of G have distinct representations. The minimum k for which G has a resolving k-decomposition is its decomposition dimension $\text{dec}(G)$. In this paper, decomposition dimension of Cartesian product of paths, cycles and stars is studied.

Keywords: Decomposition dimension; graph decomposition; Cartesian product.

Mathematics Subject Classification 2020: 05C05, 05C70

1. Introduction

Let $G = (V, E)$ be a finite, undirected, simple, connected graph. A decomposition of a graph G is a collection of subgraphs of G, none of which has isolated vertices, whose edge sets provide a partition of $E(G)$. A decomposition of G into k subgraphs is a k-decomposition of G. A decomposition $\mathscr{D} = \{G_1, G_2, \ldots, G_k\}$ is ordered if the ordering $(G_1, G_2, \ldots, G_k)$ has been imposed on $\mathscr{D}$. If each subgraph G_i of $\mathscr{D}$ is isomorphic to a graph H, then $\mathscr{D}$ is said to be an H-decomposition of G.

For edges $e, f \in E(G)$, the distance $d(e, f)$ between e and f is the minimum non-negative integer k for which there exists a sequence $e = e_0, e_1, e_2, \ldots, e_k = f$ of edges of G such that e_i and e_{i+1} are adjacent for $i = 0, 1, \ldots, k-1$. The following definitions are from [5]. If $d(g, e) \neq d(g, f)$, then the edge $g \in E(G)$

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ON THE MEAN SQUARE AVERAGE OF DIRICHLET *L*-FUNCTION OVER CHARACTERS OF ODD PARITY IN A SPECIAL CASE

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Abstract: Evaluating the mean square averages of the Dirichlet L-functions over Dirichlet characters χ of the same parity is an active problem in number theory. Here we explicitly evaluate $\sum_{\chi \text{ odd}} L(3,\chi)$ using certain trigonometric sums and Bernoulli polynomials and express the sum in terms of the Euler totient function ϕ and the Jordan totient function J_s.

Keywords: L-functions, trigonometric sums, Jordan totient function, Euler totient function, mean square averages, Gauss sum, Ramanujan sum, Bernoulli numbers.

1. Introduction

Let k be a natural number $\geqslant 3$. A Dirichlet character χ is defined to be odd if $\chi(-1) = -1$ and even if $\chi(-1) = 1$. The Dirichlet L-function $L(s,\chi)$ is defined by the infinite series $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ where $s \in \mathbb{C}$ with $Re\,(s) > 1$. It is an important function in number theory especially due to its connection with the Rieman zeta function $\zeta(s)$. For rational integer r, the problem of computing exact values of

$$\sum_{\substack{\chi \bmod k \\ \chi(-1)=(-1)^r}} |L(r,\chi)|^2 \tag{1}$$

and thus finding the mean square averages of this sum has been attempted in various cases by many.

In 1982, Walum [15] gave an exact formula for the sum (1) with $r = 1$. Louboutin ([6]) computed the sum of $|L(1,\chi)|^2$ over all odd primitive Dirichlet characters modulo k. See [4, Chapter 6] for the definition of primitivity of Dirichlet characters. In [7], Louboutin gave an exact formula for the sum of $|L(1,\chi)|^2$ over all odd Dirichlet characters in terms of the prime divisors of k and the Euler totient function ϕ. He mainly used the orthogonality properties of characters and

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Research Article

Gallai-Ramsey number for rainbow S_3

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Abstract

For the given graphs G and H, and for a positive integer k, the Gallai-Ramsey number is denoted by $gr_k(G : H)$ and is defined as the minimum integer n such that every coloring of the complete graph K_n using at most k colors contains either a rainbow copy of G or a monochromatic copy of H. The k-color Ramsey number for G, denoted by $R_k(G)$, is the minimum integer n such that every coloring of K_n using at most k colors contains a monochromatic copy of G in some color. Let S_n be the star graph on n edges and let P_n be the path graph on n vertices. Denote by S_n^+ the graph obtained from S_n by adding an edge between any two pendant vertices. Let T_{n+2} be the tree on $n+2$ vertices obtained from S_n by subdividing one of its edges. In this paper, we consider $gr_k(S_3 : H)$, where $H \in \{S_n, S_n^+, P_n, T_{n+2}\}$, and obtain its relation with $R_2(H)$ and $R_3(H)$. We also obtain 3-color Ramsey numbers for S_n, S_n^+, and T_{n+2}.

Keywords: Gallai-Ramsey number; coloring; rainbow copy; monochromatic copy.

2020 Mathematics Subject Classification: 05C15, 05C55, 05D10.

1. Introduction

In this paper, edge-colorings of finite simple graphs are considered. Throughout this paper, by coloring we mean edge-coloring. For an integer $k \geq 1$, let $\mathcal{C} : E(G) \to \{1, 2, ..., k\}$ be a k-coloring of a graph G. Thus, $\mathcal{C}$ partitions the edge set of G, $E(G)$, into k sets $C_1, C_2, \cdots, C_k$, where C_i consists of those edges of G that are colored with color i. Note that $\mathcal{C}$ need not be a proper coloring. The color i is represented at a vertex v if some edge incident with v has color i. A coloring of a graph is called monochromatic if all edges are colored the same, and a coloring is called rainbow if all edges are colored differently. Given a graph G, the k-color Ramsey number for G, denoted by $R_k(G)$, is the minimum integer n such that every coloring of the complete graph K_n using at most k colors contains a monochromatic copy of G in some color. For the given graphs G and H, and for a positive integer k, the Gallai-Ramsey number, denoted by $gr_k(G : H)$, is defined as the minimum integer n such that every coloring of K_n using at most k colors contains either a rainbow copy of G or a monochromatic copy of H. For any graph H, the inequality $gr_k(G : H) \leq R_k(H)$ holds.

In 1967, Gallai [4] investigated the structures of rainbow triangle-free (i.e., there is no rainbow K_3) colorings of complete graphs and proved the following result. In honor of Gallai's work, a coloring of a complete graph G is said to be Gallai coloring if G is rainbow triangle-free.

Theorem 1.1. [4] *In any Gallai colored complete graph G, $V(G)$ can be partitioned into non-empty sets $H_1, H_2, \cdots, H_l$, with $l \geq 2$, such that there are at most two colors between the parts, and there is only one color on the edges between every pair of parts.*

In recent years, many results on Gallai-Ramsey numbers concerning the case when G is a triangle have been reported [2, 3, 8]. However, Gallai-Ramsey numbers for other choices of G have been much less studied. In [6], the authors proved the following theorem for $G = P_4$ and posed a conjecture when $G = P_5$.

Theorem 1.2. [6] *For any graph H with no isolated vertices, $gr_k(P_4 : H) = R_2(H)$ except when $H = P_3$ and $k \geq 3$, in which case $gr_k(P_4 : P_3) = 5$.*

Conjecture 1.1. [6] *For any graph H with no isolated vertices, $gr_k(P_5 : H) = R_3(H)$.*

Gyárfás et al. [5] proved the next result concerning 3-color Ramsey numbers of paths, which was conjectured by Faudree and Schelp in [1].

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Theorem 1.3. [5] *For sufficiently large* n, $R_3(P_n) = \begin{cases} 2n-1 & \text{if } n \text{ is odd}, \\ 2n-2 & \text{if } n \text{ is even}. \end{cases}$

In this paper, we consider $gr_k(G : H)$ for rainbow S_3 and monochromatic stars, paths and some extensions of stars. Few results are known for the case when $G = S_3$ and finding this number for a path is a fundamental work. Let S_n be the star on $n+1$ vertices and n edges. Denote by S_n^+ the graph obtained from S_n by adding an edge between any two pendant vertices. Let P_n be the path on n vertices and T_{n+2} be the tree on $n+2$ vertices obtained from the star S_n with one edge subdivided. Let $V = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of the complete graph K_n. For any non-empty subset V' of V, the subgraph of K_n whose vertex set is V' and edge set is the set of those edges of K_n that have both ends in V' is called the subgraph of K_n induced by V', denoted by $K_n[V']$.

2. Main results

In this section, 3-color Ramsey numbers for S_n, S_n^+, and T_{n+2} are obtained. Also, in this section, it is shown that for all $k \geq 3$, the inequality $R_2(H) \leq gr_k(S_3 : H) \leq R_3(H)$ holds when $H \in \{S_n, S_n^+, P_n, T_{n+2}\}$. It is clear that $gr_2(S_3 : H) = R_2(H)$.

Theorem 2.1. $R_3(S_n) = 3n - 1$.

Proof. To prove $R_3(S_n) \geq 3n-1$, it is enough to show that there exist a 3-coloring of K_{3n-2} that does not contain a monochromatic copy of S_n. Let us take $G_1 = K_{3n-2}[\{v_1, v_2, \cdots, v_{n-1}\}]$, $G_2 = K_{3n-2}[\{v_n, v_{n+1}, \cdots, v_{2n-2}\}]$ and $G_3 = K_{3n-2}[\{v_{2n-1}, v_{2n}, \cdots, v_{3n-3}\}]$. Color the edges of G_i with color i where $i = 1, 2, 3$. The edge $e = uv$ is colored with color 1 if $u \in G_2, v \in G_3$, with color 2 if $u \in G_1, v \in G_3$ and with color 3 if $u \in G_1, v \in G_2$. Now, the edge $e = uv_{3n-2}$ is assigned color 1 if $u \in G_1$, color 2 if $u \in G_2$ and color 3 if $u \in G_3$. Under this coloring each vertex in K_{3n-2} is represented by color i where $i = 1, 2, 3$, at most $n-1$ times. Thus, K_{3n-2} does not contain a monochromatic copy of S_n. Hence, $R_3(S_n) \geq 3n-1$.

Now, consider any 3-coloring of K_{3n-1} and let v be any vertex in K_{3n-1}. Since $deg(v) = 3n-2$, at least n edges incident with v must be of same color giving a monochromatic copy of S_n. Thus, $R_3(S_n) \leq 3n-1$ and hence $R_3(S_n) = 3n-1$. □

Theorem 2.2. $R_3(T_{n+2}) = 3n$.

Proof. The lower bound can be proved by showing that there exist a 3-coloring of K_{3n-1} that does not contain a monochromatic copy of T_{n+2}. Let $G_1 = K_{3n-1}[\{v_1, v_2, \cdots, v_{n-1}\}]$, $G_2 = K_{3n-1}[\{v_n, v_{n+1}, \cdots, v_{2n-2}\}]$ and $G_3 = K_{3n-1}[\{v_{2n-1}, v_{2n}, \cdots, v_{3n-3}\}]$. Color the edges of G_i and the edges $w_iv_{3n-2}, w_iv_{3n-1}, w_i \in V(G_i)$ with color i where $i = 1, 2, 3$. The edge $e = uv$ is colored with color 1 if $u \in G_2, v \in G_3$, with color 2 if $u \in G_1, v \in G_3$ and with color 3 if $u \in G_1, v \in G_2$. Assign color 1 for the edge $v_{3n-2}v_{3n-1}$. Under this coloring K_{3n-1} does not contain a monochromatic copy of T_{n+2}. So, $R_3(T_{n+2}) \geq 3n$.

To prove the upper bound consider a 3-coloring $\mathcal{C} = \{C_1, C_2, C_3\}$ of K_{3n}. Since $deg(v_{3n}) = 3n-1$, at least n edges incident with v_{3n} must be of same color. Let $\{v_{3n}v_1, v_{3n}v_2, \cdots, v_{3n}v_n\} \subseteq C_1$. If there is an edge $v_iv_j \in C_1, 1 \leq i \leq n, n+1 \leq j \leq 3n-1$, then K_{3n} contains a monochromatic copy of T_{n+2}.

Now, suppose that each edge v_iv_j, $1 \leq i \leq n, n+1 \leq j \leq 3n-1$ belongs to C_2 or C_3. Then a monochromatic copy of T_{n+2} in K_{3n} can be obtained as follows. For $i = 1, 2, 3$, let $E_i = \{v_iv_j, n+1 \leq j \leq 3n-1\}$. Then $|E_i| = 2n-1$ and the edges of E_i are colored with color 2 or color 3. So, in each E_i, n edges are of same color. Let $E_i' \subset E_i$ be such that $|E_i'| = n$ and all edges of E_i' are of same color. Among E_1', E_2', E_3', two of the sets must have edges in same color. Suppose C_2 contains E_1' and E_2'. Then for some $r, n+1 \leq r \leq 3n-1$ there exists a vertex v_r such that the edges $v_1v_r \in E_1'$ and $v_2v_r \in E_2'$. If such a vertex v_r does not exist, then the set of n end vertices of edges in E_1' and the set of n end vertices of edges in E_2' are disjoint. This implies that there exist $2n$ vertices in the set $\{v_j, n+1 \leq j \leq 3n-1\}$, which is not possible. Then $E_1' \cup \{v_rv_2\}$ will give a monochromatic copy of T_{n+2} in K_{3n} in color 2. Thus, $R_3(T_{n+2}) \leq 3n$. Hence, $R_3(T_{n+2}) = 3n$. □

Lemma 2.1. *Any* 2*-coloring of* K_{2k+1} *contains a monochromatic copy of* S_k^+.

Proof. Consider a 2-coloring $\mathcal{C} = \{C_1, C_2\}$ of K_{2k+1}. Suppose there is a vertex v in K_{2k+1} such that $k+1$ edges incident with v have same color. Let $\{v_{2k+1}v_1, v_{2k+1}v_2, \cdots, v_{2k+1}v_{k+1}\} \subseteq C_1$. If there exist some edge v_iv_j, $1 \leq i < j \leq k+1$, in C_1, K_{2k+1} contains a monochromatic copy of S_k^+ in color 1. Suppose such an edge does not exist. This will imply that every edge of the induced subgraph $G' = K_{2k+1}[\{v_1, v_2, \cdots, v_{k+1}\}]$ is in C_2. Thus, G' and hence K_{2k+1} contains a monochromatic copy of S_k^+ in color 2.

Now, suppose there is no vertex in K_{2k+1} incident with $k+1$ edges in same color. Then every vertex is incident with exactly k edges in C_1 and k edges in C_2. Let $\{v_{2k+1}v_1, v_{2k+1}v_2, \cdots, v_{2k+1}v_k\} \subseteq C_1$. As in the case above if there exist some edge v_iv_j, $1 \leq i < j \leq k$, in C_1, K_{2k+1} contains a monochromatic copy of S_k^+ in color 1. If not, then every edge of $K_{2k+1}[\{v_1, v_2, \cdots, v_k\}]$ is colored with color 2. Since v_k is incident to k edges that are colored with color 2, there exist an edge v_kv_t in C_2, where $k+1 \leq t \leq 2k$. Thus, $\{v_kv_i,\ 1 \leq i \leq k-1\} \cup \{v_kv_t\} \cup \{v_1v_2\}$ is a monochromatic copy of S_k^+ in color 2 contained in K_{2k+1}. □

Theorem 2.3. $R_3(S_n^+) = 5n+1$.

Proof. To prove the lower bound consider K_{5n}. Let $G_1 = K_{5n}[\{v_1, v_2, \cdots, v_n\}]$, $G_2 = K_{5n}[\{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}]$, $G_3 = K_{5n}[\{v_{2n+1}, v_{2n+2}, \cdots, v_{3n}\}]$, $G_4 = K_{5n}[\{v_{3n+1}, v_{3n+2}, \cdots, v_{4n}\}]$ and $G_5 = K_{5n}[\{v_{4n+1}, v_{4n+2}, \cdots, v_{5n}\}]$. Assign color 1 to the edges in G_i for $1 \leq i \leq 5$. All edges in K_{5n} between G_1 and G_2, G_1 and G_3, G_2 and G_4, G_3 and G_5, G_4 and G_5 are colored with color 2. Remaining edges in K_{5n} are colored with color 3. This gives a 3-coloring of K_{5n} which contains a monochromatic copy of S_n but does not contain a monochromatic copy of S_n^+. So, $R_3(S_n^+) \geq 5n+1$.

Consider a 3-coloring $\mathcal{C} = \{C_1, C_2, C_3\}$ of K_{5n+1}. Since $deg(v_{5n+1}) = 5n$ and for $n \geq 3$, $3(n+2) \leq 5n$, at least $n+2$ edges incident with v_{5n+1} must have same color. Now, either $n+2$ or $n+1$ must be an odd number and let that odd number be $2k+1$ for some integer k. Let $\{v_{5n+1}v_1, v_{5n+1}v_2, \cdots, v_{5n+1}v_{n+2}\} \subseteq C_1$. If there is an edge $v_iv_j \in C_1$, $1 \leq i < j \leq n+2$, then K_{5n+1} contains a monochromatic copy of S_n^+.

If there is no such edge, $G_1 = K_{5n+1}[\{v_1, v_2, \cdots, v_{2k+1}\}]$ must be 2-colored. Also G_1 is isomorphic to the complete graph K_{2k+1}. Then by Lemma 2.1, G_1 contains a monochromatic copy of S_k^+ in color 2 and let $\{v_1, v_2, \cdots, v_k, v_{k+1}\}$ be the vertices of $S_k^+ \subseteq G_1$, where v_{k+1} is the hub vertex. If there are $n-k$ edges in $K_{5n+1} \setminus S_k^+$ in color 2 incident with v_{k+1}, then K_{5n+1} contains a monochromatic copy of S_n^+.

Otherwise at most $n-k-1$ edges in color 2 are incident with v_{k+1}. So, at least $4n+1$ edges incident with v_{k+1} are in C_1 or C_3. Among these, $2n+1$ edges must be in C_t where $t = 1$ or 3. Let $\{v_{k+1}v_{5n}, v_{k+1}v_{5n-1}, \cdots, v_{k+1}v_{3n}\} \subseteq C_t$ and let $G_2 = K_{5n+1}[\{v_{3n}, v_{3n+1}, \cdots, v_{5n}\}]$. If there is an edge v_rv_s, $3n \leq r < s \leq 5n$ in color t, then K_{5n+1} contains a monochromatic copy of S_n^+.

If there is no such edge, then G_2 is 2-colored. Then by Lemma 2.1, there is a monochromatic copy of S_n^+ in G_2 and hence in K_{5n+1}. So, $R_3(S_n^+) \leq 5n+1$. Hence, $R_3(S_n^+) = 5n+1$. □

Lemma 2.2. $gr_k(S_3 : H) \geq R_2(H)$, *where* $H \in \{S_n, T_{n+2}, P_n, S_n^+\}$.

Proof. By the definition of $R_2(H)$, there is a 2-coloring of K_m where $m = R_2(H) - 1$ which has no monochromatic copy of H. Since only two colors are used, K_m cannot have a rainbow copy of S_3. So, $gr_k(S_3 : H) \geq R_2(H)$. □

Theorem 2.4. $gr_k(S_3 : S_n) = 2n$.

Proof. Consider K_{2n-1}. Color the edges of the induced subgraphs $G_1 = K_{2n-1}[\{v_1, v_2, \cdots, v_{n-1}\}]$ and $G_2 = K_{2n-1}[\{v_n, v_{n+1}, \cdots, v_{2n-2}\}]$ with color 1 and color 2 respectively. Use color 3 for the edges between G_1 and G_2. The edges between the vertices of G_1 and v_{2n-1} are colored with color 1 and those between G_2 and v_{2n-1} are colored with color 2. Now, every vertex of K_{2n-1} are two colored and hence there does not exist a rainbow S_3 in K_{2n-1}. Only a monochromatic S_{n-1} could be obtained with the above coloring. Hence, $gr_k(S_3 : S_n) \geq 2n$.

Let $\mathcal{C}$ be a k-coloring of K_{2n}. If there is a vertex in K_{2n} represented by at least 3 colors, a rainbow copy of S_3 is obtained. If not, $\mathcal{C}$ is such that every vertex of K_{2n} is at most 2-colored. Let v be a vertex of K_{2n}. Since degree of v is $2n-1$, n edges incident with v must be of same color. These n edges gives a monochromatic copy of S_n in K_{2n}. Hence, $gr_k(S_3 : S_n) \leq 2n$. Thus, $gr_k(S_3 : S_n) = 2n$. □

Theorem 2.5. $R_2(S_n) \leq gr_k(S_3 : S_n) \leq R_3(S_n)$.

Proof. From Lemma 2.2, Theorem 2.1, and Theorem 2.4, the result follows. □

Theorem 2.6. $gr_k(S_3 : T_{n+2}) = 2n+1$.

Proof. Consider the complete graph K_{2n}. Color the edges of the induced subgraph $G_1 = K_{2n}[\{v_1, v_2, \cdots, v_{n+1}\}]$ with color 1. Now, color all the edges except the edge v_1v_{n+1} of the induced subgraph $G_2 = K_{2n}[\{v_{n+1}, v_{n+2}, \cdots, v_{2n}, v_1\}]$ with color 2. Use color 3 for the edges connecting the vertices of $G_1 \backslash \{v_1, v_{n+1}\}$ and $G_2 \backslash \{v_1, v_{n+1}\}$. Only a monochromatic S_n is obtained with the above coloring in color 1 and color 2. In color 3 a monochromatic S_{n-1} is obtained. So, $gr_k(S_3 : T_{n+2}) \geq 2n+1$.

Let $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ be a k-coloring of K_{2n+1}. If there is a vertex in K_{2n+1} represented by at least 3 colors, a rainbow copy of S_3 is obtained. If not, $\mathcal{C}$ is such that every vertex of K_{2n+1} is at most 2-colored. Since degree of v_{2n+1} is $2n$, at least n edges incident with v_{2n+1} must be of same color. Without loss of generality, let the edges $v_{2n+1}v_i, 1 \leq i \leq n$ be in C_1. Let $W_1 = \{v_1, v_2, \cdots, v_n\}$ and $W_2 = \{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}$. If there is an edge in C_1 with one end in W_1 and other end in W_2, a monochromatic copy of T_{n+2} in color 1 exist. If not, each $v_1w, w \in W_2$ must be in C_2. Now, if each $v_2w, w \in W_2$ is in C_2, $\{v_1w : w \in W_2\} \cup \{v_2v_{2n}\}$ gives a monochromatic copy of T_{n+2} in color 2. If each $v_2w, w \in W_2$ is in

C_3, v_3v_{2n} must be in C_2 or C_3. If $v_3v_{2n} \in C_2$, $\{v_1w : w \in W_2\} \cup \{v_3v_{2n}\}$ gives a monochromatic copy of T_{n+2} in color 2. Otherwise $\{v_2w : w \in W_2\} \cup \{v_3v_{2n}\}$ gives a monochromatic copy of T_{n+2} in color 3. Hence, $gr_k(S_3 : T_{n+2}) \leq 2n + 1$. Thus, $gr_k(S_3 : T_{n+2}) = 2n + 1$. □

Theorem 2.7. $R_2(T_{n+2}) \leq gr_k(S_3 : T_{n+2}) \leq R_3(T_{n+2})$.

Proof. From Lemma 2.2, Theorem 2.2, and Theorem 2.6, the result follows. □

Theorem 2.8. $gr_k(S_3 : S_n^+) = 2n + 1$, *where* S_n^+ *is obtained from* S_n *by adding an edge between any two pendant vertices.*

Proof. Consider the complete graph K_{2n}. Color the edges of the induced subgraphs $G_1 = K_{2n}[\{v_1, v_2, \cdots, v_n\}]$ and $G_2 = K_{2n}[\{v_{n+1}, v_{n+2}, \cdots, v_{2n}\}]$ with color 1 and color 2 respectively. Use color 3 for the edges between G_1 and G_2. Now, every vertex of K_{2n} are two colored and hence there does not exist a rainbow S_3 in K_{2n}. Only a monochromatic S_n could be obtained with the above coloring. Hence, $gr_k(S_3 : S_n^+) \geq 2n + 1$.

Let $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ be a k-coloring of K_{2n+1}. If there is a vertex in K_{2n+1} represented by at least 3 colors, a rainbow copy of S_3 is obtained. If not, $\mathcal{C}$ is such that every vertex of K_{2n+1} is at most 2-colored.

Assume that there is a vertex in K_{2n+1} incident with $n + 1$ edges and all these edges have the same color. Let $\{v_1v_{2n+1}, v_2v_{2n+1}, \cdots, v_{n+1}v_{2n+1}\} \subseteq C_1$ and let $G_1 = K_{2n+1}[\{v_1, v_2, \cdots, v_{n+1}\}]$. If there is an edge in C_1 which belongs to G_1, we get a monochromatic copy of S_n^+ in color 1. If not, every edge of G_1 must be in C_2. Then G_1 contains a monochromatic copy of S_n^+ in color 2.

Now, assume that there does not exist such a vertex. Then each vertex must have n edges in one color and n edges in another color. Let these edges be $v_1v_{2n+1}, v_2v_{2n+1}, \cdots, v_nv_{2n+1}$ in C_1 and let $G_2 = K_{2n+1}[\{v_1, v_2, \cdots, v_n\}]$. If there is an edge in C_1 which belongs to G_2, a monochromatic copy of S_n^+ is obtained in color 1. If not, every edge of G_2 is in C_2. Now, v_n is incident with $n - 1$ edges in C_2. Since v_n must have n edges in color 2, there must exist an edge v_rv_n in C_2 for some r, $n + 1 \leq r \leq 2n$. Then $v_1v_n, v_2v_n, \cdots, v_{n-1}v_n, v_rv_n$ and v_1v_2 gives a monochromatic copy of S_n^+ in color 2. Hence, $gr_k(S_3 : S_n^+) \leq 2n + 1$. So, $gr_k(S_3 : S_n^+) = 2n + 1$. □

Theorem 2.9. $R_2(S_n^+) \leq gr_k(S_3 : S_n^+) \leq R_3(S_n^+)$.

Proof. From Lemma 2.2, Theorem 2.3, and Theorem 2.8, the result follows. □

Theorem 2.10. *For* $n \geq 3$, $R_2(P_n) \leq gr_k(S_3 : P_n) \leq R_3(P_n)$.

Proof. The lower bound is clear from Lemma 2.2. When at most three colors are used, from the definition of $R_3(P_n)$ it is clear that $gr_k(S_3 : P_n) \leq R_3(P_n)$. Suppose at least four colors are used. The upper bound is established by applying induction on n. $R_3(P_3) = 5$ (from [7]) and in any k-coloring of K_5 without a rainbow S_3, each vertex of K_5 must be incident with at most 2 colors. Since $deg(v) = 4 \ \forall \ v \in K_5$, at least two edges incident to v must be of same color, which is a monochromatic copy of P_3. Thus, $gr_k(S_3 : P_3) \leq R_3(P_3)$.

Suppose that $gr_k(S_3 : P_{n-1}) \leq R_3(P_{n-1})$. The inequality $gr_k(S_3 : P_n) \leq R_3(P_n)$ is to be proved. Let $m = R_3(P_n)$. It is enough to show that any k-coloring of K_m contains a rainbow copy of S_3 or a monochromatic copy of P_n. Let $\mathcal{C} = \{C_1, C_2, \cdots, C_k\}$ be a k-coloring of K_m. Suppose that K_m does not contain a rainbow copy of S_3. Then at most two colors are represented at each vertex of K_m. Here it will be proved that K_m contains a monochromatic copy of P_n. Observe that $R_3(P_{n-1}) \leq R_3(P_n)$. Then from the induction hypothesis we get $gr_k(S_3 : P_{n-1}) \leq R_3(P_n) = m$. Since K_m does not contain a rainbow copy of S_3, it must contain a monochromatic copy of P_{n-1}. Without loss of generality, let $v_1v_2 \cdots v_{n-1}$ be a monochromatic copy of P_{n-1} in color 1. Let $G_1 = K_m[\{v_2, v_3, \cdots, v_{n-2}\}]$ and $G_2 = K_m[\{v_n, v_{n+1}, \cdots, v_m\}]$. If there is an edge v_1w or $v_{n-1}w$ for some $w \in G_2$ in color 1, then K_m contains a monochromatic copy of P_n. If not, for all $w \in G_2$ the edges $v_1w \notin C_1$ and $v_{n-1}w \notin C_1$. Since $v_1v_2 \in C_1$, all the edges v_1w, $w \in G_2$ must belong to C_i for some fixed $i, i \geq 2$ (otherwise a rainbow copy of S_3 is obtained at v_1). Same argument holds for $v_{n-1}w$, $w \in G_2$. Consider the following cases.

Case 1. For all $w \in G_2$, $v_1w \in C_2$ and $v_{n-1}w \in C_3$.
The colors, color 2 and color 3 are represented at each vertex of G_2, color 1 and color 2 at v_1, color 1 and color 3 at v_{n-1} (see Figure 1). The edges $v_nu, u \in G_1$ must be in C_2 or C_3 and hence two colors are represented at each vertex of G_1. Thus, two colors are represented at each vertex of K_m using color 1, color 2 or color 3. So, in this case $k \geq 4$ is not possible (If $k \geq 4$, then K_m contains a rainbow copy of S_3). When $k = 3$ the existence of a monochromatic copy of P_n in K_m is assured by the definition of $R_3(P_n)$, since $m = R_3(P_n)$ is the smallest integer such that every coloring of K_m with at most 3 colors will contain a monochromatic copy of P_n.

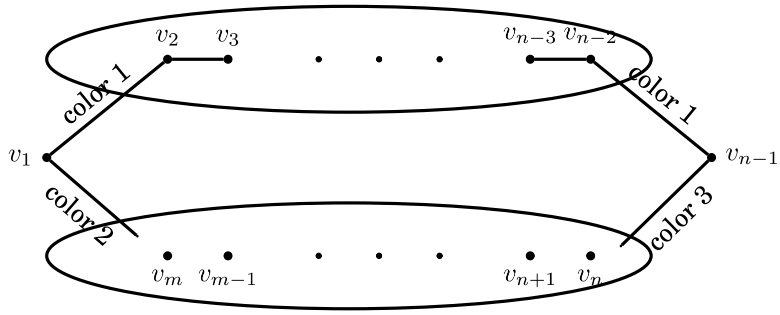

Figure 1: Case 1 of the proof of Theorem 2.10.

Case 2. For all $w \in G_2$, both v_1w and $v_{n-1}w$ are in C_2.

Subcase 2.1. For some $i \geq 3$, K_m has an edge in C_i with one end in G_1 and the other in G_2.
Without loss of generality suppose that K_m has an edge in C_3 with one end in G_1 and the other in G_2. Let v_rv_s belong to C_3 where $v_r \in G_1, v_s \in G_2$. Then color 1 and color 3 are represented at v_r, color 2 and color 3 are represented at v_s (see Figure 2). So, each edge $v_su, u \in G_1$ must be in C_2 or C_3 (otherwise a rainbow copy of S_3 is obtained at v_s) and the edges $v_rw, w \in G_2$ must be in C_1 or C_3 (otherwise a rainbow copy of S_3 is obtained at v_r). Then two colors are represented at each vertex of K_m. So, as in case 1, $k \geq 4$ is not possible and when $k = 3$, by definition of $R_3(P_n)$ there exist a monochromatic copy of P_n in K_m.

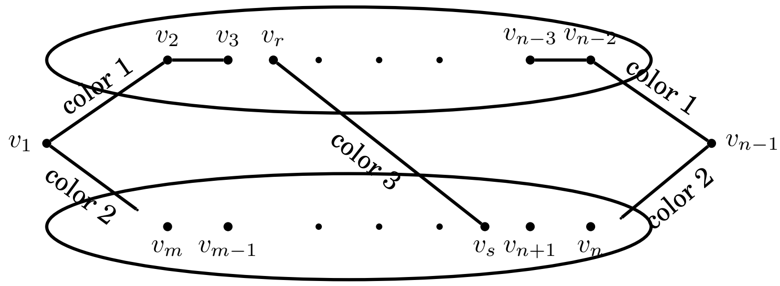

Figure 2: Subcase 2.1 of the proof of Theorem 2.10.

Subcase 2.2. For any $i, i \geq 3$, K_m has no edge in C_i with one end in G_1 and the other in G_2.
Since at least four colors are used to color the edges of K_m, C_3 is non empty. From the supposition of this subcase, the edges having color 3 must belong to G_1 or G_2 (or both). Then two cases are to be considered.

Subcase 2.2.1. Suppose G_2 contains an edge that belongs to C_3.
Let v_rv_s be the edge of G_2 that belongs to C_3 (see Figure 3).

Claim 1. Two colors, color 1 and color 2 are represented at every vertex of $V(G_1) \cup \{v_1, v_{n-1}\}$.
From the supposition of case 2, $v_1v_r \in C_2$, so color 2 is represented at v_r. Thus, two colors, color 2 and color 3 are represented at v_r. Consider the edges $v_ru, u \in G_1$. Then v_ru must have color 2 or color 3 (otherwise a rainbow copy of S_3 is obtained at v_r). From the supposition of subcase 2.2, $v_ru \notin C_3$ and hence $v_ru \in C_2$ for all $u \in G_1$. Since $u \in G_1$, color 1 is represented at u. Thus, two colors, color 1 and color 2, are represented at each vertex of G_1. So, any edge from G_1 to G_2 must be in C_1 or C_2 (otherwise a rainbow copy of S_3 is obtained). Also color 1 and color 2 are represented at the vertices v_1, v_{n-1} (from the supposition of case 2). Thus, two colors, color 1 and color 2 are represented at the vertices of $V(G_1) \cup \{v_1, v_{n-1}\}$.

Let $W = \{w \in G_2 : uw \in C_2 \; \forall \; u \in G_1\}$. Since $v_ru \in C_2$ for all $u \in G_1$, $v_r \in W$ and hence $W \neq \emptyset$. Consider the set $K_m \backslash W$.

Claim 2. Two colors, color 1 and color 2 are represented at every vertex of $K_m \backslash W$.
$V(K_m \backslash W) = V(G_1) \cup \{v_1, v_{n-1}\} \cup V(G_2 \backslash W)$. If $G_2 \backslash W = \emptyset$, then $V(K_m \backslash W) = V(G_1) \cup \{v_1, v_{n-1}\}$. Hence, from claim 1, color 1 and color 2 are represented at every vertex of $V(K_m \backslash W)$. Suppose $G_2 \backslash W \neq \emptyset$. Let x be a vertex of $G_2 \backslash W$. Since $x \in G_2$, color 2 is represented at x and since $x \notin W$, there exist some $u \in G_1$ such that $ux \notin C_2$. So, $ux \in C_1$, since any edge from G_1 to G_2 must be in C_1 or C_2. Thus, two colors, color 1 and color 2, are represented at each vertex of $G_2 \backslash W$. Also from claim 1, color 1 and color 2 are represented at each vertex of G_1 and at the vertices v_1, v_{n-1}. Hence, color 1 and color 2 are

represented at every vertex of $K_m \backslash W$. Thus, claim 2 is proved.

So, every edge that is not colored using color 1 or color 2 must be in $K_m[W]$ (otherwise a rainbow copy of S_3 is obtained at a vertex of $K_m \backslash W$).

i) Let $|W| \geq \lfloor \frac{n}{2} \rfloor$. Then $v_1 w_1 v_2 w_2 \ldots v_{\frac{n}{2}} w_{\frac{n}{2}}$ is a monochromatic copy of P_n in color 2 when n is even and $v_1 w_1 v_2 w_2 \ldots v_{\lfloor \frac{n}{2} \rfloor} w_{\lfloor \frac{n}{2} \rfloor} v_{\lfloor \frac{n}{2} \rfloor + 1}$ is a monochromatic copy of P_n in color 2 when n is odd, where $w_i \in W$ for $i \geq 1$.

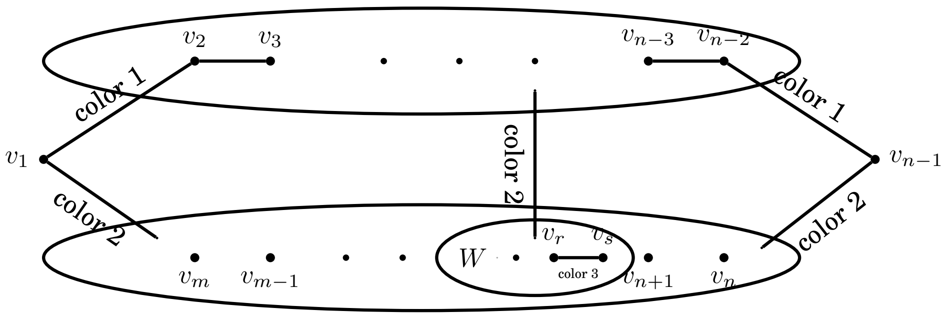

Figure 3: Subcase 2.2.1 of the proof of Theorem 2.10.

ii) Let $|W| < \lfloor \frac{n}{2} \rfloor$. It will be proved that K_m contains a monochromatic copy of P_n in color 1 or color 2. For that construct a 3-coloring of K_m from $\mathcal{C}$ using color 1, color 2 and color 3. Under $\mathcal{C}$ every edge of $E(K_m) \backslash E(W)$ is in color 1 or color 2 (from claim 2). Recolor the edges of $K_m[W]$ alone using color 3. This recoloring gives a new 3-coloring, $\mathcal{C}'$, of K_m. Then, from the definition of $R_3(P_n)$, K_m contains a monochromatic copy of P_n under $\mathcal{C}'$. All the edges of K_m having color 3 under $\mathcal{C}'$ belongs to $K_m[W]$ and hence if the monochromatic copy of P_n under $\mathcal{C}'$ is in color 3, then it must be contained in $K_m[W]$. But $|W| < \lfloor \frac{n}{2} \rfloor$. So, the monochromatic copy of P_n under $\mathcal{C}'$ is not in $K_m[W]$. This implies that the monochromatic copy of P_n in K_m under $\mathcal{C}'$ is not in color 3 and hence it is either in color 1 or in color 2. Without loss of generality suppose that the monochromatic copy of P_n under $\mathcal{C}'$ is in color 1 and let $e_1 e_2 \ldots e_{n-1}$ be the edges in P_n. It is to be noted that every edge of K_m having color 1 or color 2 under $\mathcal{C}'$ had the same color under $\mathcal{C}$. Then these e_i's will have color 1 in K_m under $\mathcal{C}$ and hence a monochromatic copy of P_n in color 1 is obtained under $\mathcal{C}$.

Subcase 2.2.2. Suppose that G_2 does not contain an edge that belongs to C_3.
From the supposition in subcase 2.2, every edge in C_3 must be in G_1. Let $v_r v_s$ be an edge in G_1 that belong to C_3. Then color 1 and color 3 is represented at v_r. So, the edges $v_r w, w \in G_2$ must be in C_1 or C_3 (otherwise a rainbow copy of S_3 is obtained at v_r). From the supposition of subcase 2.2 $v_r w$ cannot have color 3. So, for all $w \in G_2$, $v_r w$ is in color 1. Thus, two colors, color 1 and color 2, are represented at each vertex in G_2 and at the vertices v_1, v_{n-1}. Recolor G_1 with color 3 to obtain a 3-coloring $\mathcal{C}'$ of K_m. Then from the definition of $R_3(P_n)$, K_m contains a monochromatic copy of P_n under $\mathcal{C}'$. Since $|G_1| < n$, this monochromatic copy of P_n is not in color 3 and hence it is either in color 1 or in color 2. Then the same monochromatic copy of P_n in K_m under $\mathcal{C}'$ can be obtained under $\mathcal{C}$. Thus, in all cases $gr_k(S_3 : P_n) \leq R_3(P_n)$. □

Remark 2.1. *Let us consider an example for which strict inequality holds in Theorem 2.10. We have $R_3(P_3) = 5$. But, $gr_k(S_3 : P_3) = 4$. Consider a k-coloring of K_4 that does not contain a rainbow S_3. Then at most two colors are represented at each vertex of K_4. Since the degree of each vertex of K_4 is three, there exist at least two edges in the same color incident with each vertex of K_4, giving a monochromatic copy of P_3. So, $gr_k(S_3 : P_3) \leq 4$. Now, the complete graph on three vertices, C_3 does not contain a rainbow copy of S_3 or a monochromatic copy of P_3 in any 3-coloring. Hence, $gr_k(S_3 : P_3) = 4$.*

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Decomposition dimension of corona product of some classes of graphs

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Abstract

For an ordered k-decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph $G = (V, E)$, the $\mathcal{D}$-representation of an edge e is the k-tuple $\gamma(e/\mathcal{D}) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k))$, where $d(e, G_i)$ represents the distance from e to G_i. A decomposition $\mathcal{D}$ is resolving if every two distinct edges of G have distinct representations. The minimum k for which G has a resolving k-decomposition is its decomposition dimension $dec(G)$. In this paper, the decomposition dimension of corona product of the path P_n and cycle C_n with the complete graphs K_1 and K_2 are determined.

Key words: *Decomposition dimension, Corona product, Path, Cycle.*

Mathematical Subject Classification Codes: *05C38, 05C70*

1. Introduction

Let $G = (V, E)$ be a finite undirected connected graph without loops or multiple edges. A decomposition of a graph G is a collection of subgraphs of G, none of which have isolated vertices, whose edge sets provide a partition of $E(G)$. A decomposition of G into k subgraphs is a k-decomposition of G. A decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ is ordered if the ordering $(G_1, G_2, \ldots, G_k)$ has been imposed on $\mathcal{D}$. If each subgraph G_i of $\mathcal{D}$ is isomorphic to a graph H, then $\mathcal{D}$ is said to be an H-decomposition of G.

For edges $e, f \in E(G)$, the distance $d(e, f)$ between e and f is the minimum non negative integer k for which there exists a sequence $e = e_0, e_1, e_2, \ldots, e_k = f$ of edges of G such that e_i and e_{i+1} are adjacent for $i = 0, 1, \ldots, k-1$. For an edge e of G and a subgraph F of G, $d(e, F) = min\{d(e, f), f \in E(F)\}$. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ be an ordered k-decomposition of G. The $\mathcal{D}$-representation of an edge e is the k-tuple $\gamma(e/\mathcal{D}) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k))$, where $d(e, G_i)$ represents the distance from e to G_i. We call $\mathcal{D}$ a resolving k-decomposition if for any pair of edges e and f, there exists some index i such that $d(e, G_i) \neq d(f, G_i)$. The minimum k for which G has a resolving k-decomposition is its decomposition dimension $dec(G)$. These concepts were introduced by Chartrand et.al in [1]. It is further studied in [2,3,8].

The concepts of resolving set and minimum resolving set have appeared in the literature previously. Slater introduced and studied these ideas with a different terminology 'locating set' in [9]. Harary and Melter [4] discovered these concepts independently. Later these concepts were rediscovered by Johnson in [5]. Chartrand et.al [1] proved that $dec(G) \geq 3$ for all connected graphs G that are not paths and for a tree T of order n and diameter d, $dec(T) \leq n - d + 1$. M. Hagita, A. Kundgen and D. B. West [3] used probabilistic methods to obtain upper bounds for decomposition dimension of complete graphs and regular graphs. H. Enomoto and T. Nakamigawa [2] established a lower bound for decomposition dimension of graphs using the maximum degree of G. They proved that for any graph G, $dec(G) \geq \lceil log_2 \Delta(G) \rceil + 1$. Reji T. and Ruby R. studied about decomposition dimension of cartesian product of graphs in [6].

The corona product, $G_1 \odot G_2$ of two graphs G_1 (with n_1 vertices and m_1 edges) and G_2 (with n_2 vertices and m_2 edges) is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2, and then joining the ith vertex of G_1 with an edge to every vertex in the ith copy of G_2.

Metric dimension and partition dimension, which distinguishes the vertices of a graph using distance, of corona product of graphs are studied in [7,10].

2. Main Results

Define $\alpha_i^+ : \mathbf{R}^n \to \mathbf{R}^n$ by $\alpha_i^+(x_1, \ldots, x_i, \ldots, x_n) = (x_1, \ldots, x_i + 1, \ldots, x_n)$ and $\alpha_i^- : \mathbf{R}^n \to \mathbf{R}^n$ by $\alpha_i^-(x_1, \ldots, x_i, \ldots, x_n) = (x_1, \ldots, x_i - 1, \ldots, x_n)$

Theorem 1. $dec(P_n \odot K_1) = \begin{cases} 2 & \text{if } n = 2 \\ 3 & \text{if } n \geq 3 \end{cases}$

Proof. **Case 1:** $n = 2$
The corona product of the path P_2 and the complete graph K_1, $P_2 \odot K_1$ is the path P_4. Hence $dec(P_2 \odot K_1) = 2$.

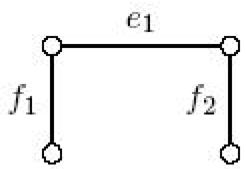

Figure 1. $P_2 \odot K_1$.

Case 2: $n \geq 3$
The corona product of the path P_n and the complete graph K_1, $P_n \odot K_1$ is also known as the n-centipede graph. Let $v_1, v_2, \ldots, v_n$ be the n vertices and $e_1, e_2, \ldots, e_{n-1}$ be the $n-1$ edges of the path P_n. Label the edges joining the vertex v_i in P_n and K_1 as $f_i, 1 \leq i \leq n$.

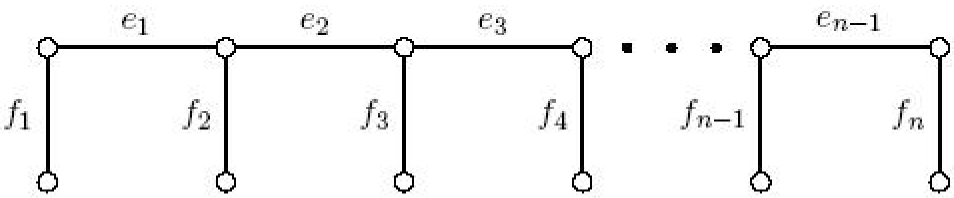

Figure 2. $P_n \odot K_1$.

Consider the decomposition $\mathcal{D} = \{G_1, G_2, G_3\}$ of $P_n \odot K_1$ where $E(G_1) = \{f_1\}$, $E(G_2) = \{f_n\}$ and $E(G_3)$ consists of all other edges of $P_n \odot K_1$. Then $\gamma(f_1/\mathcal{D}) = (0, n, 1)$, $\gamma(f_n/\mathcal{D}) = (n, 0, 1)$, $\gamma(f_i/\mathcal{D}) = (i, n+1-i, 0)$, $2 \leq i \leq n-1$ and $\gamma(e_i/\mathcal{D}) = (i, n-i, 0), 1 \leq i \leq n-1$. Thus $\mathcal{D}$ is a resolving decomposition of $P_n \odot K_1$. So $dec(P_n \odot K_1) \leq 3$. Since $P_n \odot K_1$ is not a path $dec(P_n \odot K_1) \geq 3$. Hence $dec(P_n \odot K_1) = 3$. □

Theorem 2. *$dec(P_2 \odot K_2) = 3$ and $dec(P_n \odot K_2) \leq 4$, if $n \geq 3$*

Proof. **Case 1:** $n = 2$
Consider the graph $P_2 \odot K_2$. Let v_1, v_2 be the vertices of the path P_2 and e_1 be the edge joining v_1 and v_2 in P_2. For $i = 1, 2$ label the edges joining the vertex v_i in P_2 and K_2 as f_i, g_i and let h_i be the edge in K_2 adjacent to the edges f_i and g_i.

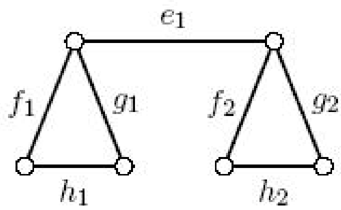

Figure 3. $P_2 \odot K_2$.

Consider the decomposition $\mathcal{D} = \{G_1, G_2, G_3\}$ of $P_2 \odot K_2$ where $E(G_1) = \{g_1\}$, $E(G_2) = \{g_2\}$ and $E(G_3)$ consists of all other edges of $P_2 \odot K_2$. Then $\gamma(g_1/\mathcal{D}) = (0, 2, 1)$, $\gamma(g_2/\mathcal{D}) = (2, 0, 1)$, $\gamma(f_1/\mathcal{D}) = (1, 2, 0)$, $\gamma(f_2/\mathcal{D}) = (2, 1, 0)$, $\gamma(h_1/\mathcal{D}) = (1, 3, 0)$, $\gamma(h_2/\mathcal{D}) = (3, 1, 0)$, $\gamma(e_1/\mathcal{D}) = (1, 1, 0)$. Thus $\mathcal{D}$ is a resolving decomposition of $P_2 \odot K_2$. So $dec(P_2 \odot K_2) \leq 3$. Since $P_2 \odot K_2$ is not a path, $dec(P_2 \odot K_2) \geq 3$. Hence $dec(P_2 \odot K_2) = 3$.

Case 2: $n \geq 3$
Consider the corona product of the path P_n and the complete graph K_2, $P_n \odot K_2$. Let $v_1, v_2, \ldots, v_n$ be the n vertices and $e_1, e_2, \ldots, e_{n-1}$ be the $n-1$ edges of the path P_n. For $i = 1, 2, \ldots, n$ label the edges joining the vertex v_i in P_n and K_2 as f_i, g_i and let h_i be the edge in K_2 adjacent to the edges f_i and g_i.

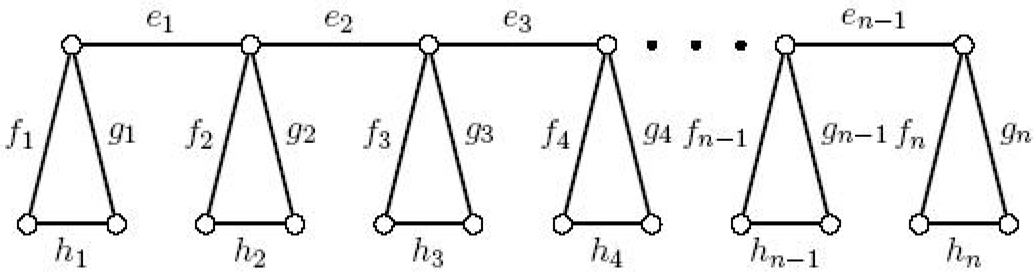

Figure 4. $P_n \odot K_2$.

Since $P_n \odot K_2$ is not a path, $dec(P_n \odot K_2) \geq 3$. Consider the decomposition $\mathcal{D} = \{G_1, G_2, G_3, G_4\}$ of $P_n \odot K_2$ where $E(G_1) = \{g_1\}$, $E(G_2) = \{g_2, g_3, \ldots, g_{n-1}\}$, $E(G_3) = \{g_n\}$ and $E(G_4)$ consists of all other edges of $P_n \odot K_2$.

Then $\gamma(g_1/\mathcal{D}) = (0, 2, n, 1)$, $\gamma(g_n/\mathcal{D}) = (n, 2, 0, 1)$, $\gamma(f_1/\mathcal{D}) = (1, 2, n, 0)$, $\gamma(f_n/\mathcal{D}) = (n, 2, 1, 0)$, $\gamma(h_1/\mathcal{D}) = (1, 3, n+1, 0)$, $\gamma(h_n/\mathcal{D}) = (n+1, 3, 1, 0)$, $\gamma(e_i/\mathcal{D}) = (i, 1, n-i, 0), 1 \leq i \leq n-1$.
For $2 \leq i \leq n-1$, $\gamma(g_i/\mathcal{D}) = (i, 0, n+1-i, 1)$, $\gamma(f_i/\mathcal{D}) = (i, 1, n+1-i, 0)$, $\gamma(h_i/\mathcal{D}) = (i+1, 1, n+2-i, 0)$. Thus $\mathcal{D}$ is a resolving decomposition of $P_n \odot K_2$. So $dec(P_n \odot K_2) \leq 4$. □

Theorem 3. $dec(C_n \odot K_1) = 3$

Proof. Consider the corona product of the cycle C_n and the complete graph K_1, $C_n \odot K_1$. Let $v_1, v_2, \ldots, v_n$ be the n vertices of the path C_n and $e_1, e_2, \ldots, e_n$ be the n edges of the cycle C_n. Label the edges joining the vertex v_i in C_n and K_1 as $f_i, 1 \leq i \leq n$.

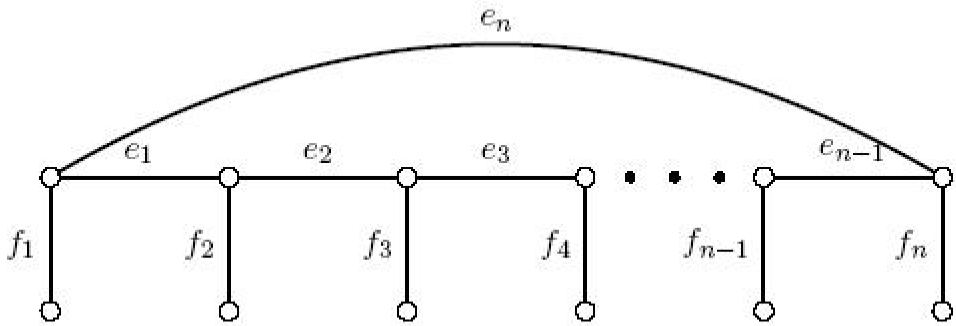

Figure 5. $C_n \odot K_1$.

Let $n \geq 3$ be any positive integer. Then $n = 3k-1$ *or* $3k$ *or* $3k+1$, where $k = 1, 2, \ldots$. Consider the decomposition $\mathcal{D} = \{G_1, G_2, G_3\}$ of $C_n \odot K_1$.

Case 1: $n = 3k - 1$

Let $E(G_1) = \{f_1, f_n, f_{n-1}, \ldots, f_{n-k+3}\}$, $E(G_2) = \{f_2, f_3, \ldots, f_{k+1}\}$ and $E(G_3)$ consists of all other edges of $C_n \odot K_1$. Then

$$\gamma(f_i/\mathcal{D}) = \begin{cases} (0,2,1) & \text{if } i = 1 \\ (i,0,1) & \text{if } 2 \leq i \leq k+1 \\ (k+1,2,0) & \text{if } i = k+2 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(f_{i-1})) & \text{if } k+3 \leq i \leq n-k+2 \\ (0,k,1) & \text{if } i = n-k+3 \\ \alpha_2^-(\gamma(f_{i-1})) & \text{if } n-k+4 \leq i \leq n \end{cases}$$

$$\gamma(e_i/\mathcal{D}) = \begin{cases} (i,1,0) & \text{if } 1 \leq i \leq k+1 \\ (k,2,0) & \text{if } i = k+2 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(e_{i-1})) & \text{if } k+3 \leq i \leq n-k+2 \\ (1,k-1,0) & \text{if } i = n-k+3 \\ \alpha_2^-(\gamma(e_{i-1})) & \text{if } n-k+4 \leq i \leq n \end{cases}$$

Case 2: $n = 3k$ *or* $3k+1$

Let $E(G_1) = \{f_1, f_n, f_{n-1}, \ldots, f_{n-k+2}\}$, $E(G_2) = \{f_2, f_3, \ldots, f_{k+1}\}$ and

$E(G_3)$ consists of all other edges of $C_n \odot K_1$.
When $n = 3k$

$$\gamma(f_i/\mathcal{D}) = \begin{cases} (0,2,1) & \text{if } i = 1 \\ (i,0,1) & \text{if } 2 \leq i \leq k+1 \\ (k+1,2,0) & \text{if } i = k+2 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(f_{i-1})) & \text{if } k+3 \leq i \leq n-k+1 \\ (0,k+1,1) & \text{if } i = n-k+2 \\ \alpha_2^-(\gamma(f_{i-1})) & \text{if } n-k+3 \leq i \leq n \end{cases}$$

$$\gamma(e_i/\mathcal{D}) = \begin{cases} (i,1,0) & \text{if } 1 \leq i \leq k+1 \\ (k,2,0) & \text{if } i = k+2 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(e_{i-1})) & \text{if } k+3 \leq i \leq n-k+1 \\ (1,k,0) & \text{if } i = n-k+2 \\ \alpha_2^-(\gamma(e_{i-1})) & \text{if } n-k+3 \leq i \leq n \end{cases}$$

When $n = 3k+1$

$$\gamma(f_i/\mathcal{D}) = \begin{cases} (0,2,1) & \text{if } i = 1 \\ (i,0,1) & \text{if } 2 \leq i \leq k+1 \\ (k+2,2,0) & \text{if } i = k+2 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(f_{i-1})) & \text{if } k+3 \leq i \leq n-k+1 \\ (0,k+1,1) & \text{if } i = n-k+2 \\ \alpha_2^-(\gamma(f_{i-1})) & \text{if } n-k+3 \leq i \leq n \end{cases}$$

$$\gamma(e_i/\mathcal{D}) = \begin{cases} (i,1,0) & \text{if } 1 \leq i \leq k+1 \\ (k+1,2,0) & \text{if } i = k+2 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(e_{i-1})) & \text{if } k+3 \leq i \leq n-k \\ (1,k+1,0) & \text{if } i = n-k+1 \\ \alpha_2^-(\gamma(e_{i-1})) & \text{if } n-k+2 \leq i \leq n \end{cases}$$

Thus $\mathcal{D}$ is a resolving decomposition of $C_n \odot K_1$. So $dec(C_n \odot K_1) \leq 3$. Since $C_n \odot K_1$ is not a path $dec(C_n \odot K_1) \geq 3$. Hence $dec(C_n \odot K_1) = 3$. □

Theorem 4. $dec(C_n \odot K_2) \leq 4$

Proof. Consider the corona product of the cycle C_n and the complete graph K_2, $C_n \odot K_2$. Let $v_1, v_2, \ldots, v_n$ be the n vertices of the path C_n and $e_1, e_2, \ldots, e_n$ be the n edges of the cycle C_n. For $i = 1, 2, \ldots, n$ label the edges joining the vertex v_i in C_n and K_2 as f_i, g_i and let h_i be the edge in K_2 adjacent to the edges f_i and g_i.

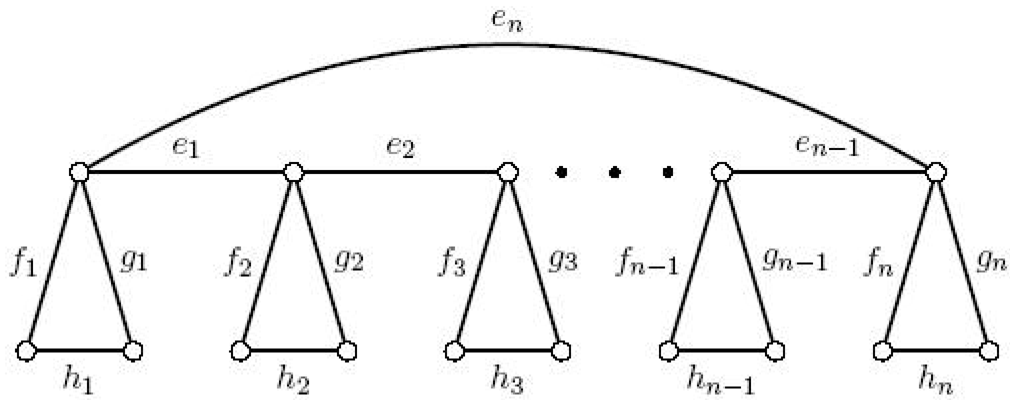

Figure 6. $C_n \odot K_2$.

Let n be any positive integer. By division algorithm there exists positive integers q, r such that $n = 3q+r$ where $r = 0$ or 1 or 2. Since $C_n \odot K_2$ is not a path, $dec(C_n \odot K_2) \geq 3$. Consider the decomposition $\mathcal{D} = \{G_1, G_2, G_3, G_4\}$ of $C_n \odot K_2$.

Case 1: $n = 3q$

Let $E(G_1) = \{g_1, g_2, \ldots, g_q\}$, $E(G_2) = \{g_{q+1}, g_{q+2}, \ldots, g_{2q}\}$, $E(G_3) = \{g_{2q+1}, g_{2q+2}, \ldots, g_n\}$ and $E(G_4)$ consists of all other edges of $C_n \odot K_2$. Then

$$\gamma(f_i/\mathcal{D}) = \begin{cases} (1, q+2-i, i+1, 0) & \text{if } 1 \le i \le q \\ (2, 1, q+1, 0) & \text{if } i = q+1 \\ (\alpha_1^+ \circ \alpha_3^-)(\gamma(f_{i-1})) & \text{if } q+2 \le i \le 2q \\ (q+1, 2, 1, 0) & \text{if } i = 2q+1 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(f_{i-1})) & \text{if } 2q+2 \le i \le n \end{cases}$$

$$\gamma(e_i/\mathcal{D}) = \begin{cases} (1, q+1-i, i+1, 0) & \text{if } 1 \le i \le q \\ (\alpha_1^+ \circ \alpha_3^-)(\gamma(e_{i-1})) & \text{if } q+1 \le i \le 2q \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(e_{i-1})) & \text{if } 2q+1 \le i \le n \end{cases}$$

$$\gamma(h_i/\mathcal{D}) = \begin{cases} (\alpha_2^+ \circ \alpha_3^+)(\gamma(f_i)) & \text{if } 1 \le i \le q \\ (\alpha_1^+ \circ \alpha_3^+)(\gamma(f_i)) & \text{if } q+1 \le i \le 2q \\ (\alpha_1^+ \circ \alpha_2^+)(\gamma(f_i)) & \text{if } 2q+1 \le i \le n \end{cases}$$

$\gamma(g_i/\mathcal{D}), 1 \le i \le n$ is obtained by replacing 1 and 0 in corresponding $\gamma(f_i/\mathcal{D})$ by 0 and 1.

Case 2: $n = 3q+1$
Let $E(G_1) = \{g_1, g_n, \ldots, g_{q+1}\}$, $E(G_2) = \{g_{q+2}, g_{q+2}, \ldots, g_{2q+1}\}$, $E(G_3) = \{g_{2q+2}, g_{2q+3}, \ldots, g_n\}$ and $E(G_4)$ consists of all other edges of $C_n \odot K_2$. Then

$$\gamma(f_i/\mathcal{D}) = \begin{cases} (1, q+3-i, i+1, 0) & \text{if } 1 \le i \le q+1 \\ (2, 1, q+1, 0) & \text{if } i = q+2 \\ (\alpha_1^+ \circ \alpha_3^-)(\gamma(f_{i-1})) & \text{if } q+3 \le i \le 2q+1 \\ (q+1, 2, 1, 0) & \text{if } i = 2q+2 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(f_{i-1})) & \text{if } 2q+3 \le i \le n \end{cases}$$

$$\gamma(e_i/\mathcal{D}) = \begin{cases} (1, q+2-i, i+1, 0) & \text{if } 1 \le i \le q \\ (1, 1, q+1, 0) & \text{if } i = q+1 \\ (\alpha_1^+ \circ \alpha_3^-)(\gamma(e_{i-1})) & \text{if } q+2 \le i \le 2q+1 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(e_{i-1})) & \text{if } 2q+2 \le i \le n \end{cases}$$

$$\gamma(h_i/\mathcal{D}) = \begin{cases} (\alpha_2^+ \circ \alpha_3^+)(\gamma(f_i)) & \text{if } 1 \le i \le q+1 \\ (\alpha_1^+ \circ \alpha_3^+)(\gamma(f_i)) & \text{if } q+2 \le i \le 2q+1 \\ (\alpha_1^+ \circ \alpha_2^+)(\gamma(f_i)) & \text{if } 2q+2 \le i \le n \end{cases}$$

$\gamma(g_i/\mathcal{D}), 1 \le i \le n$ is obtained by replacing 1 and 0 in corresponding $\gamma(f_i/\mathcal{D})$ by 0 and 1.

Case 3: $n = 3q+2$
Let $E(G_1) = \{g_1, g_n, \ldots, g_{q+1}\}$, $E(G_2) = \{g_{q+2}, g_{q+3}, \ldots, g_{2q+2}\}$, $E(G_3) = \{g_{2q+3}, g_{2q+4}, \ldots, g_n\}$ and $E(G_4)$ consists of all other edges of $C_n \odot K_2$. Then

$$\gamma(f_i/\mathcal{D}) = \begin{cases} (1, q+3-i, i+1, 0) & \text{if } 1 \le i \le q+1 \\ (2, 1, q+2, 0) & \text{if } i = q+2 \\ (\alpha_1^+ \circ \alpha_3^-)(\gamma(f_{i-1})) & \text{if } q+3 \le i \le 2q+2 \\ (q+1, 2, 1, 0) & \text{if } i = 2q+3 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(f_{i-1})) & \text{if } 2q+4 \le i \le n \end{cases}$$

$$\gamma(e_i/\mathcal{D}) = \begin{cases} (1, q+2-i, i+1, 0) & \text{if } 1 \le i \le q+1 \\ (\alpha_1^+ \circ \alpha_3^-)(\gamma(e_{i-1})) & \text{if } q+2 \le i \le 2q+1 \\ (q+1, 1, 1, 0) & \text{if } i = 2q+2 \\ (\alpha_1^- \circ \alpha_2^+)(\gamma(e_{i-1})) & \text{if } 2q+3 \le i \le n \end{cases}$$

$$\gamma(h_i/\mathcal{D}) = \begin{cases} (\alpha_2^+ \circ \alpha_3^+)(\gamma(f_i)) & \text{if } 1 \le i \le q+1 \\ (\alpha_1^+ \circ \alpha_3^+)(\gamma(f_i)) & \text{if } q+2 \le i \le 2q+2 \\ (\alpha_1^+ \circ \alpha_2^+)(\gamma(f_i)) & \text{if } 2q+3 \le i \le n \end{cases}$$

$\gamma(g_i/\mathcal{D}), 1 \le i \le n$ is obtained by replacing 1 and 0 in corresponding $\gamma(f_i/\mathcal{D})$ by 0 and 1.

Thus $\mathcal{D}$ is a resolving decomposition of $C_n \odot K_2$. So $dec(C_n \odot K_2) \le 4$. □

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LEAST COMMON MULTIPLE OF PRODUCT GRAPHS

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Abstract A graph G without isolated vertices is a least common multiple of two graphs H_1 and H_2 if G is a smallest graph, in terms of number of edges, such that there exists a decomposition of G into edge disjoint copies of H_1 and H_2. The collection of all least common multiples of H_1 and H_2 is denoted by $LCM(H_1, H_2)$ and the size of a least common multiple of H_1 and H_2 is denoted by $lcm(H_1, H_2)$. In this paper $lcm(P_4, C_m \,\square\, P_n)$, $lcm(P_4, W_m \,\square\, P_n)$ and $lcm(P_4, W_m \,\square\, C_n)$ are determined where the product is the cartesian product.

1 Introduction

All graphs considered in this paper are assumed to be simple and to have no isolated vertices. The size of a graph G is the number of edges of G denoted by $|E(G)|$. A graph H is said to divide a graph G if there exists a set of subgraphs of G, each isomorphic to H, whose edge sets partition the edge set of G. Such a set of subgraphs is called an H-*decomposition* of G. G is said to be H-*decomposable* if G has an H- *decomposition* and write $H|G$.

A graph G is called a common multiple of two graphs H_1 and H_2 if both $H_1|G$ and $H_2|G$. A graph G is a least common multiple of H_1 and H_2 if G is a common multiple of H_1 and H_2 and no other common multiple has fewer edges. Several authors have investigated the problem of finding least common multiples of pairs of graphs H_1 and H_2; that is graphs of minimum size which are both H_1 and H_2 decomposable. The problem was introduced by Chartrand et.al in [4] and they showed that every two nonempty graphs have a least common multiple. The problem of finding the size of least common multiples of graphs has been studied for several pairs of graphs: cycles and stars [4, 13, 14], paths and complete graphs [9], pairs of cycles [8], pairs of complete graphs [3], complete graphs and a 4-cycle [2], pairs of cubes [1], complete graph and star [11] and paths and stars [7]. Pairs of graphs having a unique least common multiple were investigated by several authors [6, 12, 10]. Least common multiple of digraphs were considered in [5].

An obvious necessary condition for the existence of a graph G which is a common multiple of H_1 and H_2 is that both $|E(H_1)|$ and $|E(H_2)|$ divide $|E(G)|$. This condition is not always sufficient. Therefore, we may ask: Given two graphs H_1 and H_2, for which value of q does there exist a graph G having q edges which is a common multiple of the graphs H_1 and H_2? Adams, Bryant and Maenhaut [2] gave a complete solution to this problem in the case where H_1 is the 4-cycle and H_2 is a complete graph; Bryant and Maenhaut [3] gave a complete solution to this problem in the case where H_1 is the complete graph K_3 and H_2 is a complete graph. Thus the problem to find least common multiple of H_1 and H_2 is to find the least positive integer q such that there exists a graph G having q edges which is both H_1 and H_2 decomposable. We denote the set of all least common multiples of H_1 and H_2 by $LCM(H_1, H_2)$. The size of a least common multiple of H_1 and H_2 is denoted by $lcm(H_1, H_2)$. Since every two nonempty graphs have a least common multiple, $LCM(H_1, H_2)$ is nonempty. The number of elements in the set $LCM(H_1, H_2)$ is greater than one for many pairs of graphs. For example both P_7 and C_6 are least common multiples of P_4 and P_3.

In fact, Chartrand et.al [6] proved that for every positive integer n there exist two graphs having exactly n least common multiples. In [9] it was shown that every least common multiple of two connected graphs is connected and that every least common multiple of two 2-connected graphs is 2-connected. But this is not the case for disconnected graphs. For example if we take $H_1 = 2K_2$, $H_2 = C_5$, then $G_1 = 2C_5$ and G_2 which is the graph obtained by identifying two vertices in two copies of C_5, are in $LCM(H_1, H_2)$ of which G_1 is disconnected while G_2 is connected.

2 Main Result

The cartesian product of two graphs G and H denoted by $G \,\Box\, H$ is a graph with vertex set $V(G)\times V(H)$ for which $\{(x,u),(y,v)\}$ is an edge if $x = y$ and $\{u,v\} \in E(H)$ or $\{x,y\} \in E(G)$ and $u = v$. The graph $G \,\Box\, H$ has $|V(G)||V(H)|$ vertices and $|V(G)||E(H)| + |V(H)||E(G)|$ edges. In this section graphs that belong to $LCM(P_4, C_m \,\Box\, P_n)$, $LCM(P_4, W_m \,\Box\, P_n)$ and $LCM(P_4, W_m \,\Box\, C_n)$ are constructed and hence computed the lcm of the respective pairs of graphs. Let G^t for $t = 1,2,3$ denote the t-th copy of the graph G. Also let v^t and e^t denote a vertex and an edge in G^t.

2.1 lcm of P_4 and $C_m \,\Box\, P_n$

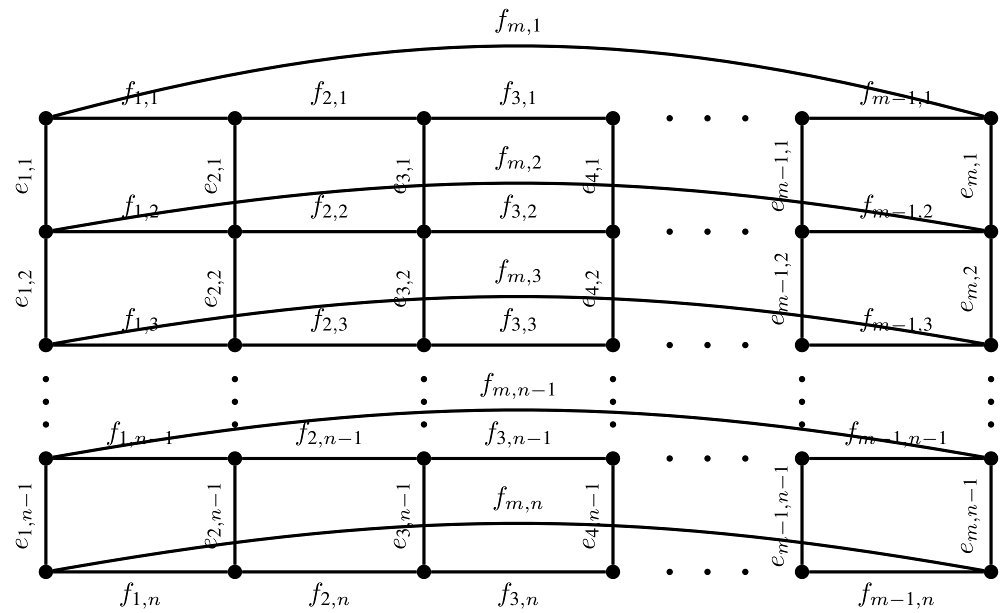

Figure 1. $\mathbf{C_m \,\Box\, P_n}$

Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be the vertices of C_m and P_n respectively. $C_m \times \{b_j\}$, $1 \le j \le n$ are the C_m-fibers and $\{a_i\} \times P_n$, $1 \le i \le m$ are the P_n-fibers in $C_m \,\Box\, P_n$. Label the vertices and edges of the j-th C_m-fiber, $C_m \times \{b_j\}$ as $\{v_{1,j}, v_{2,j}, \ldots, v_{m,j}\}$, $\{f_{1,j}, f_{2,j}, \ldots, f_{m,j}\}$ and that of the i-th P_n-fiber, $\{a_i\} \times P_n$ as $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$, $\{e_{i,1}, e_{i,2}, \ldots, e_{i,n-1}\}$.

Theorem 2.1. $lcm(P_4, C_m \,\Box\, P_n) = \begin{cases} 2mn - m & \text{if } m \equiv 0 \pmod 3 \text{ or } n \equiv 2 \pmod 3 \\ 6mn - 3m & \text{otherwise} \end{cases}$

Proof. Least common multiple of P_4 and $C_m \,\Box\, P_n$ is the number of edges in the graph of least size that is both P_4-decomposable and $C_m \,\Box\, P_n$-decomposable. We consider various cases for m and n in modulo 3 and will construct in each case a graph of least size that is both P_4-decomposable and $C_m \,\Box\, P_n$-decomposable.

Case 1: $n = 2$, $m \in \mathbb{N}$, $m \ge 3$

The graph $G = C_m \,\Box\, P_2$ has $3m$ edges. A P_4-decomposition of G is given by the following copies of P_4: $(f_{i,1}, e_{i,1}, f_{i,2})$, $1 \le i \le m$. Thus G is P_4-decomposable and hence

$$lcm(P_4, C_m \,\Box\, P_2) = 3m.$$

Case 2: $m = 3, n \in \mathbb{N},\ n \geq 3$

In this case $G = C_3 \,\square\, P_n$, which has $6n - 3$ edges. A P_4-decomposition of G is obtained as follows:

$$\{(f_{1,j}, e_{2,j}, f_{2,j+1}), 1 \leq j \leq n-1\}, \{(e_{1,j}, f_{3,j}, e_{3,j-1}), 2 \leq j \leq n-1\},$$

$$(e_{1,1}, f_{3,1}, e_{3,1}), \qquad (f_{1,n}, f_{3,n}, e_{3,n-1})$$

Thus G is P_4-decomposable and hence $lcm(P_4, C_3 \,\square\, P_n) = 6n - 3$.

Case 3: $m = 3k,\ k \geq 2$

Subcase 3.1: $n = 3l,\ l \geq 1$

The graph $G = C_{3k} \,\square\, P_{3l}$ has $3k(3l-1) + (3l)(3k)$ edges and hence $|E(G)| \equiv 0 \pmod 3$. The $3l - 1$ edges of the i-th P_n-fiber of G, where $1 \leq i \leq m$, together with the edge $f_{i,n}$ of the n-th C_m-fiber makes a P_{3l+1}, which is P_4-decomposable. For $1 \leq j \leq n-1$, the j-th C_m-fiber contains $3k$ edges and hence it is P_4-decomposable. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k} \,\square\, P_{3l}) = 3k(3l-1) + (3l)(3k)$.

Subcase 3.2: $n = 3l + 1,\ l \geq 1$

In this case $G = C_{3k} \,\square\, P_{3l+1}$ and $|E(G)| = 3k(3l) + (3l+1)(3k) \equiv 0 \pmod 3$. Here each C_m-fiber has $3k$ edges and each P_n-fiber has $3l$ edges and hence every C_m-fiber and P_n-fiber are P_4-decomposable. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k} \,\square\, P_{3l+1}) = 3k(3l) + (3l+1)(3k)$.

Subcase 3.3: $n = 3l + 2,\ l \geq 1$

Here $G = C_{3k} \,\square\, P_{3l+2}$ and it has $3k(3l+1)+(3l+2)(3k)$ edges which is a multiple of three. The j-th C_m-fiber, where $1 \leq j \leq n-2$, has $3k$ edges and hence it is P_4-decomposable. The first $3l$ edges of the i-th P_n-fiber, where $1 \leq i \leq m$ makes a P_{3l+1}, which is P_4-decomposable. Consider the edges of the $(n-1)$-th and n-th C_m-fibers and the edges $\{e_{i,n-1}, 1 \leq i \leq m\}$. Then $\{(f_{i,n-1}, e_{i,n-1}, f_{i,n}),\ 1 \leq i \leq m\}$ gives a copy of P_4 for each i. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k} \,\square\, P_{3l+2}) = 3k(3l+1) + (3l+2)(3k)$.

Case 4: $m = 3k + 1,\ k \geq 1$

Subcase 4.1: $n = 3l,\ l \geq 1$

The graph $G = C_{3k+1} \,\square\, P_{3l}$ has $(3k+1)(3l-1)+(3l)(3k+1)$ edges and hence $|E(G)| \equiv 2 \pmod 3$. The first $3k$ edges of the j-th C_m-fiber, where $1 \leq j \leq n-1$, makes a P_{3k+1}, which is P_4-decomposable. The $3l - 1$ edges of the i-th P_n-fiber, where $2 \leq i \leq m-1$, together with the edge $f_{i-1,n}$ of the n-th C_m-fiber makes a P_{3l+1}, which is P_4-decomposable. Now $\{(e_{1,j}, f_{m,j}, e_{m,j}),\ 1 \leq j \leq n-1\}$ gives a copy of P_4 for each j. The edges $\{f_{m-1,n}, f_{m,n}\}$ are left out.

Take three copies of G namely G^1, G^2, G^3 and each copy has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{1,n}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{m-1,n}$ with the vertex $v^3_{m-1,n}$. The left out edges $\{f^t_{m-1,n}, f^t_{m,n}; t = 1,2,3\}$ in the three copies of G will make a P_7 in H, which is P_4-decomposable. Thus H is P_4-decomposable and hence $lcm(P_4, C_{3k+1} \,\square\, P_{3l}) = 3((3k+1)(3l-1) + (3l)(3k+1))$.

Subcase 4.2: $n = 3l + 1,\ l \geq 1$

In this case $G = C_{3k+1} \,\square\, P_{3l+1}$ which has $(3k+1)(3l) + (3l+1)(3k+1)$ edges and hence $|E(G)| \equiv 1 \pmod 3$. The first $3k$ edges of the j-th C_m-fiber, where $1 \leq j \leq n$, makes a P_{3k+1}, which is P_4-decomposable. For $2 \leq i \leq m-1$, the i-th P_n-fiber, has $3l$ edges and hence it is P_4-decomposable. Now $\{(e_{1,j}, f_{m,j}, e_{m,j}),\ 1 \leq j \leq n-1\}$ gives a copy of P_4 for each j. The edge $f_{m,n}$ is left out.

Take three copies of G namely G^1, G^2, G^3 and each copy has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{m,n}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{m,n}$ with the vertex $v^3_{1,n}$. The left out edges $\{f^t_{m,n}; t = 1,2,3\}$ in the three copies of G will make a P_4 in H. Thus H is P_4-decomposable and hence $lcm(P_4, C_{3k+1} \,\square\, P_{3l+1}) = 3((3k+1)(3l) + (3l+1)(3k+1))$.

Subcase 4.3: $n = 3l + 2,\ l \geq 1$

Here $G = C_{3k+1} \,\square\, P_{3l+2}$ and $|E(G)| = (3k+1)(3l+1)+(3l+2)(3k+1)$, which is a multiple of three. The first $3k$ edges of the j-th C_m-fiber, where $1 \leq j \leq n-2$, makes a P_{3k+1}, which is P_4-decomposable. The first $3l$ edges of the i-th P_n-fiber, where $2 \leq i \leq m-1$ makes a P_{3l+1}, which is P_4-decomposable. $\{(e_{1,j}, f_{m,j}, e_{m,j}),\ 1 \leq j \leq n-1\}$ gives a copy of P_4 for each j. Consider the edges of the $(n-1)$-th and n-th C_m-fibers and the edges $\{e_{i,n-1}, 1 \leq i \leq m\}$. Then

$\{(f_{i,n-1}, e_{i,n-1}, f_{i,n}),\ 1 \le i \le m\}$ gives a copy of P_4 for each i. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k+1} \square P_{3l+2}) = (3k+1)(3l+1) + (3l+2)(3k+1)$.

Case 5: $m = 3k+2,\ k \ge 1$

Subcase 5.1: $n = 3l,\ l \ge 1$

For the graph $G = C_{3k+2} \square P_{3l}$, $|E(G)| = (3k+2)(3l-1) + (3l)(3k+2) \equiv 1 \pmod 3$. The $3k+2$ edges of the j-th C_m-fiber, where $1 \le j \le n-1$, together with the edge $e_{m,j}$ of the m-th P_n-fiber, makes $3k+3$ edges, which is P_4-decomposable. The $3l-1$ edges of the i-th P_n-fiber, where $1 \le i \le m-1$, together with the edge $f_{i,n}$ of the n-th C_m-fiber makes a P_{3l+1}, which is P_4-decomposable. The edge $f_{m,n}$ is left out.

Take three copies of G namely G^1, G^2, G^3 and each copy has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{m,n}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{m,n}$ with the vertex $v^3_{1,n}$. The left out edges $\{f^t_{m,n}; t = 1,2,3\}$ in the three copies of G will make a P_4 in H. Thus H is P_4-decomposable and hence $lcm(P_4, C_{3k+2} \square P_{3l}) = 3((3k+2)(3l-1) + (3l)(3k+2))$.

Subcase 5.2: $n = 3l+1,\ l \ge 1$

In this case $G = C_{3k+2} \square P_{3l+1}$ which has $(3k+2)(3l) + (3l+1)(3k+2)$ edges and hence $|E(G)| \equiv 2 \pmod 3$. The $3k+2$ edges of the j-th C_m-fiber, where $1 \le j \le n-1$, together with the edge $e_{m,j}$ of the m-th P_n-fiber, makes $3k+3$ edges, which is P_4-decomposable. For $1 \le i \le m-1$, the i-th P_n-fiber, has $3l$ edges and hence it is P_4-decomposable. The first $3k$ edges of the n-th C_m-fiber makes a P_{3k+1}, which is P_4-decomposable. The edges $\{f_{m-1,n}, f_{m,n}\}$ are left out.

Take three copies of G namely G^1, G^2, G^3 and each copy has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{1,n}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{m-1,n}$ with the vertex $v^3_{m-1,n}$. The left out edges $\{f^t_{m-1,n}, f^t_{m,n}; t = 1,2,3\}$ in the three copies of G will make a P_7 in H, which is P_4-decomposable. Thus H is P_4-decomposable and hence $lcm(P_4, C_{3k+2} \square P_{3l+1}) = 3((3k+2)(3l) + (3l+1)(3k+2))$.

Subcase 5.3: $n = 3l+2,\ l \ge 1$

The graph $G = C_{3k+1} \square P_{3l+2}$ has $(3k+2)(3l+1) + (3l+2)(3k+2)$ edges, which is a multiple of three. The $3k+2$ edges of the j-th C_m-fiber, where $1 \le j \le n-2$, together with the edge $e_{m,j}$ of the m-th P_n-fiber, makes $3k+3$ edges, which is P_4-decomposable. The first $3l$ edges of the i-th P_n-fiber, where $1 \le i \le m-1$ makes a P_{3l+1}, which is P_4-decomposable. Consider the edges of the $(n-1)$-th and n-th C_m-fibers and the edges $\{e_{i,n-1}, 1 \le i \le m\}$. Then $\{(f_{i,n-1}, e_{i,n-1}, f_{i,n}),\ 1 \le i \le m\}$ gives a copy of P_4 for each i. Thus G is P_4-decomposable and hence $lcm(P_4, C_{3k+2} \square P_{3l+2}) = (3k+2)(3l+1) + (3l+2)(3k+2)$. □

Theorem 2.2. *$C_m \square P_n$ is P_4-decomposable if and only if $m \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$.*

2.2 lcm of P_4 and $W_m \square P_n$

Let W_m denote the wheel graph of order m, which contains a cycle C_{m-1} and a vertex called hub, which is adjacent to every vertex of C_{m-1}. $|E(W_m)| = 2m-2$. Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be the vertices of W_m and P_n respectively, where a_m is the hub vertex of W_m. $W_m \times \{b_j\}$, $1 \le j \le n$ are the W_m-fibers and $\{a_i\} \times P_n$, $1 \le i \le m$ are the P_n-fibers in $W_m \square P_n$. Label the vertices and edges of the j-th W_m-fiber, $W_m \times \{b_j\}$ as $\{v_{1,j}, v_{2,j}, \ldots, v_{m,j}\}$, $\{f_{1,j}, f_{2,j}, \ldots, f_{m-1,j}, g_{1,j}, g_{2,j}, \ldots, g_{m-1,j}\}$ where $\{f_{1,j}, f_{2,j}, \ldots, f_{m-1,j}\}$ are the edges of the cycle in the j-th W_m-fiber and $\{g_{1,j}, g_{2,j}, \ldots, g_{m-1,j}\}$ are the edges connecting the hub and the vertices of the cycle in the j-th W_m-fiber. The vertices and edges of the i-th P_n-fiber, $\{a_i\} \times P_n$ are labelled as $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$ and $\{e_{i,1}, e_{i,2}, \ldots, e_{i,n-1}\}$ respectively.

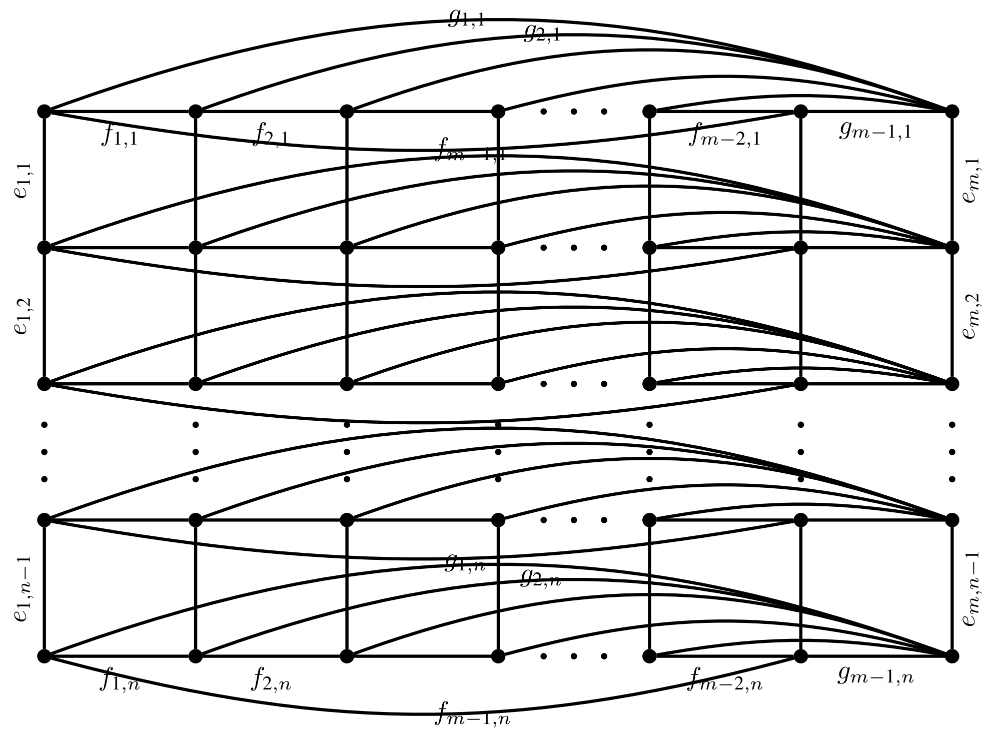

Figure 2. $\mathbf{W_m \,\Box\, P_n}$

Theorem 2.3. $lcm(P_4, W_m \,\Box\, P_n) = \begin{cases} 3mn - 2n - m & \textit{if } 2m + n \equiv 0 \pmod 3 \\ 3(3mn - 2n - m) & \textit{otherwise} \end{cases}$

Proof. Let $P^{'}$ be the path $v_{1,1}f_{1,1}v_{2,1}f_{2,1}\dots f_{m-2,1}v_{m-1,1}g_{m-1,1}v_{m,1}$, which is contained in the first W_m-fiber, $P^{''}$: $v_{m,1}e_{m,1}v_{m,2}e_{m,2}\dots v_{m,n-1}e_{m,n-1}v_{m,n}$, the m-th P_n-fiber and $P^{'''}$: $v_{m,n}g_{m-1,n}v_{m-1,n}f_{m-2,n}\dots v_{2,n}f_{1,n}v_{1,n}$, the path contained in the last W_m-fiber.

Let $G = W_m \,\Box\, P_n$. Then $|E(G)| = m(n-1) + n(2m-2) = 3mn - 2n - m$. Consider the edges of $G^* = (W_m \,\Box\, P_n)\backslash\{P^{'}, P^{''}, P^{'''}\}$. Copies of P_4 are obtained as follows :
For a fixed j, $1 \le j \le n-2$, $\{(g_{i,j}, e_{i,j}, f_{i,j+1}),\ 1 \le i \le m-2\}$, $\{(f_{m-1,j}, e_{m-1,j}, g_{m-1,j+1})\}$,

$$\{(g_{i,n-1}, e_{i,n-1}, g_{i,n}),\ 1 \le i \le m-2\}, (f_{m-1,n-1}, e_{m-1,n-1}, f_{m-1,n}).$$

Thus G^* is P_4-decomposable. The paths $P^{'}$, $P^{''}$ and $P^{'''}$ makes the path P^* of length $2m+n-3$ in $W_m \,\Box\, P_n$. Thus $W_m \,\Box\, P_n$ is P_4-decomposable if P^* is P_4-decomposable and this happens if $2m + n \equiv 0 \pmod 3$.

If $2m + n \equiv 1$ *or* $2 \pmod 3$, take three copies of G namely G^1, G^2, G^3 and in each copy of G, the subgraph G^* has the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{1,1}$ with the vertex $v^2_{1,n}$ and the vertex $v^2_{1,1}$ with the vertex $v^3_{1,1}$. Then the path P^* in the three copies of G will make a path of length $3(2m+n-3)$ in H, which is P_4-decomposable and so is H. Thus $lcm(P_4, W_m \,\Box\, P_n) = |E(W_m \,\Box\, P_n)|$ if $2m+n \equiv 0 \pmod 3$ and $3|E(W_m \,\Box\, P_n)|$ otherwise. □

Theorem 2.4. *$W_m \,\Box\, P_n$ is P_4-decomposable if and only if* $2m + n \equiv 0 \pmod 3$.

2.3 lcm of P_4 and $W_m \,\Box\, C_n$

Let $a_1, a_2, \dots, a_m$ and $b_1, b_2, \dots, b_n$ be the vertices of W_m and C_n respectively, where a_m is the hub vertex of W_m. $W_m \times \{b_j\}$, $1 \le j \le n$ are the W_m-fibers and $\{a_i\} \times C_n$, $1 \le i \le m$ are the C_n-fibers in $W_m \,\Box\, C_n$. Label the vertices and edges of the j-th W_m-fiber, $W_m \times \{b_j\}$ as in the above case of $W_m \,\square\, P_n$. The vertices and edges of the i-th C_n-fiber, $\{a_i\} \times C_n$ are labelled as $\{v_{i,1}, v_{i,2}, \ldots, v_{i,n}\}$, $\{e_{i,1}, e_{i,2}, \ldots, e_{i,n}\}$.

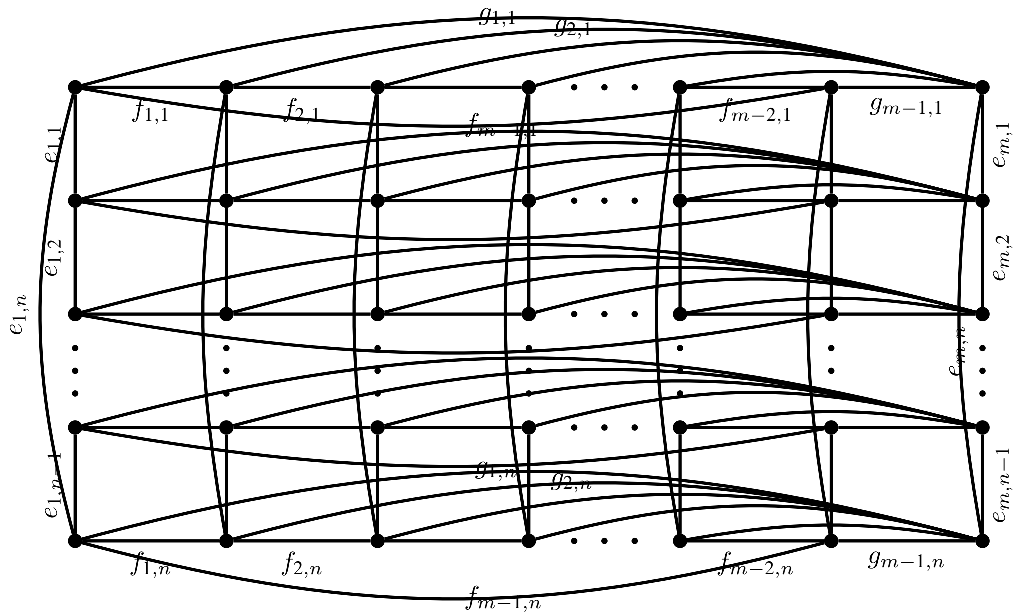

Figure 3. $\mathbf{W_m \,\square\, C_n}$

Theorem 2.5. $lcm(P_4, W_m \,\square\, C_n) = \begin{cases} 3mn - 2n & \textit{if } n \equiv 0 \pmod 3 \\ 3(3mn-2n) & \textit{otherwise} \end{cases}$

Proof. Let $G = W_m \,\square\, C_n$. Then $|E(G)| = mn + n(2m-2) = 3mn - 2n$. Copies of P_4 are obtained as follows :
For a fixed $j,\ 2 \le j \le n-2$, $\{(g_{i,j}, e_{i,j}, f_{i,j+1}),\ 1 \le i \le m-2\}$, $\{(f_{m-1,j}, e_{m-1,j}, g_{m-1,j+1})\}$,

$$\{(g_{i,1}, e_{i,n}, f_{i,n}), (f_{i,1}, e_{i,1}, f_{i,2}), (g_{i,n-1}, e_{i,n-1}, g_{i,n}); 1 \le i \le m-2\},$$

$$(f_{m-1,1}, e_{m-1,1}, g_{m-1,2}), (f_{m-1,n-1}, e_{m-1,n-1}, f_{m-1,n}), (e_{m-1,n}, g_{m-1,1}, e_{m,n})$$

The path P^* of length n consisting of the edges $\{e_{m,1}, e_{m,2}, \ldots, e_{m,n-1}, g_{m-1,n}\}$ is left out. Thus $W_m \,\square\, C_n$ is P_4-decomposable if P^* is P_4-decomposable and this happens if $n \equiv 0 \pmod 3$.

If $n \equiv 1$ *or* $2 \pmod 3$, take three copies of G namely G^1, G^2, G^3 having the above decomposition. Let H be the graph obtained by identifying the vertex $v^1_{m,1}$ with the vertex $v^2_{m,1}$ and the vertex $v^2_{m-1,n}$ with the vertex $v^3_{m-1,n}$. Then the path P^* in the three copies of G will make a path of length $3n$ in H, which is P_4-decomposable and so is H. Thus $lcm(P_4, W_m \,\square\, C_n) = |E(W_m \,\square\, C_n)|$ if $n \equiv 0 \pmod 3$ and $3|E(W_m \,\square\, C_n)|$ otherwise. □

Theorem 2.6. *$W_m \,\square\, C_n$ is P_4-decomposable if and only if $n \equiv 0 \pmod 3$.*

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